جامعة تكريت – كلية التربية للبنات – قسم الرياضيات

المرحلة الثانية – المعادلات التفاضلية الاعتيادية

الفصل التمهيدي – طرائق التكامل

أ.د. عامر فاضل نصار
amer6767@tu.edu.iq

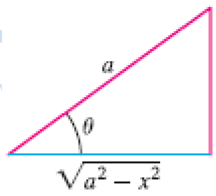
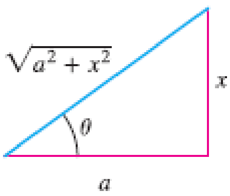
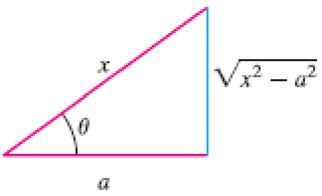
عنوان المحاضرة : الطريقة السابعة عشر : طريقة تكاملات من الأنواع

$$\int (\sec x)^m (\tan x)^n \, dx$$

الطريقة السابعة عشر Seventeenth method

طريقة التعويض بالدوال المثلثية Trigonometric substitutions method

نستخدم طريقة التعويض بالدوال المثلثية عندما نستبدل متغير التكامل بدالة مثلثية. أكثر البدائل شيوعًا هي التعويضات $x = a\sin\theta$, $x = a\sec\theta$, $x = a\tan\theta$. هذه التعويضات فعالة في تحويل التكاملات التي تتضمن تكاملات من الانواع $\sqrt{a^2 - x^2}$, $a^2 - x^2$, $\sqrt{a^2 + x^2}$, $a^2 + x^2$, $\sqrt{x^2 - a^2}$, $x^2 - a^2$ التي تحولها الى تكاملات يمكننا إيجادها مباشرة وفق الجدول التالي

ت	الدوال	الفرضيات	الاشكال
1	$\sqrt{a^2 - x^2}$, $a^2 - x^2$	$x = a\sin\theta, \quad a > 0$	
2	$\sqrt{a^2 + x^2}$, $a^2 + x^2$	$x = a\tan\theta, \quad a > 0$	
3	$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad a > 0$	

الإجراء الخاص بالتعويضات المثلثية

1. نختار الفرضية المناسبة عن x ، ونحسب التفاضلي dx ، ونحدد قيمة θ بدلالة x .
2. نعوض الفرضية في السؤال المراد حله ، ثم نبسّط النتائج جبريًا.
3. نكامل بالنسبة الى المتغير θ .
4. نرسم المثلث المطلوب للحالة .
5. نستبدل المتغيرات في نتيجة التكامل إلى المتغير الأصلي x.

Example (71): Evaluate $I = \dfrac{dx}{\sqrt{4+x^2}}$

Let $x = a\tan\theta \quad \Rightarrow \quad x = 2\tan\theta \quad \Rightarrow \quad dx = 2\sec^2\theta d\theta$

$x^2 = 4\tan^2\theta$

$$I = \int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4+4\tan^2\theta}} = \int \frac{2\sec^2\theta d\theta}{2\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \sec\theta d\theta$$

$$= \ln(\sec\theta + tan\theta) + c = \ln(\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}) + c$$

Example (72): Evaluate $I = \dfrac{dx}{\sqrt{4-x^2}}$

Let $x = a\sin\theta \quad \Rightarrow \quad x = 2\sin\theta \quad \Rightarrow \quad dx = 2\cos\theta d\theta$

$x^2 = 4\sin^2\theta$

$$I = \int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta d\theta}{2\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta d\theta}{\cos\theta} = \int d\theta = \theta = \sin^{-1}\frac{x}{2}$$

Example (73): Evaluate $I = \dfrac{x^2 dx}{\sqrt{9-x^2}}$

Let $x = a\sin\theta \quad \Rightarrow \quad x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta d\theta$

$x^2 = 3\sin^2\theta$

$$I = \int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{(9\sin^2\theta)(3\cos\theta)d\theta}{\sqrt{9-9\sin^2\theta}} = \int \frac{(9\sin^2\theta)(3\cos\theta)d\theta}{3\sqrt{1-\sin^2\theta}} = \int \frac{(9\sin^2\theta)(\cos\theta)d\theta}{\cos\theta}$$

$$I = 9\int \sin^2\theta d\theta = 9\int \frac{1-\cos 2\theta}{2} d\theta = 9\int (\frac{1}{2} - \frac{\cos 2\theta}{2})d\theta = \frac{9}{2}\int d\theta - \frac{9}{2}\int \cos 2\theta d\theta$$

$$I = \frac{9}{2}\theta - \frac{9}{2}\frac{1}{2}\int \cos 2\theta(2)d\theta = \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + c = \frac{9}{2}\theta - \frac{9}{4}(2\sin\theta\cos\theta) + c$$

$$I = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{9}{2}(\frac{x}{3}.\frac{\sqrt{9-x^2}}{3}) + c$$

Example (74): Evaluate $I = \displaystyle\int \frac{dx}{(x^2+1)^2}$

Let $x = a\tan\theta \quad \Rightarrow \quad x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta d\theta \quad \& \quad x^2 = \tan^2\theta$

$$I = \int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta d\theta}{(\tan^2\theta+1)^2} = \int \frac{\sec^2\theta d\theta}{(\sec^2\theta)^2} = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta d\theta = \int (\frac{1}{2} + \frac{1}{2}cos2\theta)d\theta$$

$$= \frac{1}{2}\theta - \sin 2\theta + c = \frac{1}{2}\tan^{-1}x - \sin(2\tan^{-1}x) + c$$

Example (75): Evaluate $I = \displaystyle\int \frac{x}{(x^2-6x+1)^3}dx$

$$I=\frac{1}{2}\int\frac{2x}{(x^2-6x+1)^3}dx=\frac{1}{2}\int\frac{2x-6+6}{(x^2-6x+1)^3}dx=\frac{1}{2}\int\frac{2x-6}{(x^2-6x+1)^3}dx+3\int\frac{1}{(x^2-6x+1)^3}dx$$

$$=\frac{1}{2}\int(x^2-6x+1)^{-3}(2x-6)dx+3\int\frac{1}{(x^2-6x+1)^3}dx=\frac{1}{2}\cdot\frac{(x^2-6x+1)^{-2}}{-2}+3\int\frac{1}{(x^2-6x+1)^3}dx$$

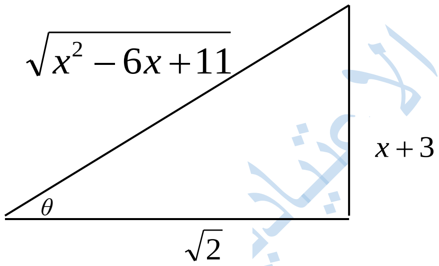
Let $I_1=\int\frac{dx}{(x^2-6x+1)^3}=\int\frac{dx}{[(x-3)^2+2]^3}$

لإيجاد I_1 ، نستخدم الفرضيات التالية من المثلث التالي.

Let $x-3=\sqrt{2}\tan\theta \Rightarrow dx=\sqrt{2}\sec^2\theta$

$(x-3)^2=2\tan^2\theta \Rightarrow (x-3)^2+2=2\tan^2\theta+2$

$\Rightarrow (x-3)^2+2=2(\tan^2\theta+1)=2\sec^2\theta$



$$I_1=\int\frac{dx}{[(x-3)^2+2]^3}=\int\frac{\sqrt{2}\sec^2\theta}{[2\sec^2\theta]^3}d\theta=\int\frac{\sqrt{2}\sec^2\theta}{8\sec^6\theta}d\theta=\frac{\sqrt{2}}{8}\int\frac{\sec^2\theta}{\sec^6\theta}d\theta=\frac{1}{4\sqrt{2}}\int\frac{1}{\sec^4\theta}d\theta$$

$$I_1=\frac{1}{4\sqrt{2}}\int\cos^4\theta d\theta$$

الآن نستخدم الصيغة

$$\int(\cos x)^n dx=\frac{(\cos x)^{n-1}\sin x}{n}+\frac{n-1}{n}\int(\cos x)^{n-2}dx$$

$$I_1=\frac{1}{4\sqrt{2}}\left\{\frac{(\cos\theta)^3\sin\theta}{4}+\frac{3}{4}\int\cos^2\theta d\theta\right\}=\frac{(\cos\theta)^3\sin\theta}{16\sqrt{2}}+\frac{3}{16\sqrt{2}}\int\cos^2\theta d\theta$$

$$I_1=\frac{(\cos\theta)^3\sin\theta}{16\sqrt{2}}+\frac{3}{16\sqrt{2}}\left\{\frac{\cos\theta\sin\theta}{2}+\frac{1}{2}\int d\theta\right\}=\frac{(\cos\theta)^3\sin\theta}{16\sqrt{2}}+\frac{3}{16\sqrt{2}}\left\{\frac{\cos\theta\sin\theta}{2}+\frac{\theta}{2}\right\}$$

$$I=\frac{1}{2}\cdot\frac{(x^2-6x+1)^{-2}}{-2}+3I_1=\frac{(x^2-6x+1)^{-2}}{-4}+\frac{3(\cos\theta)^3\sin\theta}{16\sqrt{2}}+\frac{9}{16\sqrt{2}}\left\{\frac{\cos\theta\sin\theta}{2}+\frac{\theta}{2}\right\}$$

$$I=\frac{(x^2-6x+1)^{-2}}{-4}+\frac{3}{16\sqrt{2}}\left(\frac{x-3}{\sqrt{x^2-6x+11}}\right)^3\left(\frac{\sqrt{2}}{\sqrt{x^2-6x+11}}\right)$$

$$+\frac{9}{32\sqrt{2}}\left(\frac{x-3}{\sqrt{x^2-6x+11}}\right)\cdot\left(\frac{\sqrt{2}}{\sqrt{x^2-6x+11}}\right)+\frac{9}{32\sqrt{2}}\left\{\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right)\right\}$$

....

Exercise (7-1): Evaluate the integrals by use Trigonometric substitutions method

Exe.(35) page(297) – book al-samarrai

No.	Question	Answer
1	$\int \frac{dx}{\sqrt{x^2-4}}$	$I=\ln(\sqrt{x^2-4}+x)+c$
2	$\int \frac{dx}{\sqrt{2-5x^2}}dx$	$I=\frac{1}{\sqrt{5}}\sin^{-1}(\frac{\sqrt{5}x}{\sqrt{2}})+c$
3	$\int \frac{dx}{x^2\sqrt{4+x^2}}$	$I=-\frac{\sqrt{4+x^2}}{4x}+c$
4	$\int \frac{xdx}{\sqrt{4+x^2}}$	$I=\sqrt{4+x^2}+c$
5	$\int x\sqrt{16-x^2}dx$	$I=-\frac{(16-x^2)^{3/2}}{3}+c$
6	$\int \frac{dx}{x^2-9}$	$I=\frac{-1}{3}\tanh^{-1}(\frac{x}{3})+c$
7	$\int \frac{x^2dx}{\sqrt{4-x^2}}$	$I=2\sin^{-1}(\frac{x}{2})-\frac{x\sqrt{4-x^2}}{2}+c$
8	$\int \frac{\sqrt{9-4x^2}}{x}dx$	$I=\frac{3\ln(\sqrt{9-4x^2}-3)}{2}-\frac{3\ln(\sqrt{9-4x^2}+3)}{2}+\sqrt{9-4x^2}+c$
9	$\int \frac{\sqrt{x^2+a^2}}{x}dx$	$I=\sqrt{x^2+a^2}-a\tan^{-1}(\frac{\sqrt{x^2+a^2}}{a})+c$
10	$\int \frac{dx}{x^2+2x+5}$	$I=\frac{1}{2}\tan^{-1}(\frac{x+1}{2})+c$
11	$\int \frac{dx}{2x^2+12x+20}$	$I=\frac{1}{2}\tan^{-1}(x+3)+c$
12	$\int \frac{xdx}{\sqrt{x^2-2x+5}}$	$I=\sin^{-1}(\frac{x-1}{2})+\sqrt{x^2-2x+5}+c$ $I=\ln(x+\sqrt{(x-1)^2+4}-1)+\sqrt{x^2-2x+5}+c$
13	$\int \frac{(x-1)dx}{\sqrt{x^2-4x+3}}$	$I=\ln(\sqrt{x^2-4x+3}+x-2)+\sqrt{x^2-4x+3}+c$
14	$\int \frac{dx}{(x^2+2x+2)^3}$	$I=\frac{3}{8}\tan^{-1}(x+1)+\frac{(x+1)(3x^2+6x+8)}{8(x^2+2x+2)^2}+c$
15	$\int_0^4 \frac{dx}{\sqrt{9+x^2}}$	$I=\sinh^{-1}(\frac{x}{3})\Big]_0^4=\sinh^{-1}(\frac{4}{3})$
16	$\int_0^2 \frac{dx}{\sqrt{4+x^2}}$	$I=\sinh^{-1}(1)$
17	$\int_0^1 \frac{dx}{4-x^2}$	$I=\frac{\ln 3}{4}$

18	$\int_0^3 \frac{dx}{\sqrt{x^2-2x+5}}$	$I=\sinh^{-1}(1)+\sinh^{-1}(1/2)$
19	$\int_0^2 \frac{x^2dx}{\sqrt{2x-x^2}}$	$I=3\pi/2$

(20) Find the area of circle $x^2+y^2=r^2$

(21) Find the area between the curve $y=\frac{x^2}{(4-x^2)^{3/2}}$ and x-axis and $x=0$, $x=1$.

(22) Prove $\int \frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+c$, $a>0$

(23) Prove $\int \frac{du}{\sqrt{u^2-a^2}}=\ln\left|u+\sqrt{u^2-a^2}\right|+c$, $a>0$

Exercise (7-2): Evaluate the integrals by use Trigonometric substitutions method

Exe. (5-12) page(410) blue book

No.	Question	Answer
1	$\int \frac{dx}{\sqrt{100+x^2}}dx$	$I=\sinh^{-1}(\frac{x}{10})+c$
2	$\int \frac{dx}{\sqrt{4x^2+4x-2}}$	$I=\frac{1}{2}\ln(\sqrt{4x^2+4x-2}+2x+1)+c$
3	$\int \frac{dx}{\sqrt{1-9x^2}}$	$I=\frac{1}{3}\sin^{-1}(3x)+c$
4	$\int \frac{dx}{(2x+1)\sqrt{4x^2+4x-1}}$	$I=\frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2x^2+2x-\frac{1}{2}}\right)+c$
5	$\int \sqrt{9-x^2}dx$	$I=\frac{x\sqrt{9-x^2}}{2}+9\sin^{-1}(\frac{x}{3})+c$
6	$\int \frac{xdx}{\sqrt{4x-x^2}}$	$I=2\sin^{-1}(\frac{x-2}{2})-\sqrt{(4-x)x}+c$
7	$\int \sqrt{x^2+16}dx$ $\int \sqrt{x^2-16}dx$	$I=\frac{x\sqrt{x^2+16}}{2}+8\sin^{-1}(\frac{x}{4})+c$ $I=\frac{x\sqrt{x^2-16}}{2}-8\ln(\sqrt{x^2-16}+x)+c$
8	$\int \frac{(x+1)dx}{\sqrt{2x^2-6x+4}}$	$I=\frac{2\sqrt{x^2-3x+2}+5\tanh^{-1}(\frac{2x-3}{2\sqrt{x^2-3x+2}})}{2\sqrt{2}}+c$
9	$\int \frac{dx}{x^2\sqrt{x^2-9}}$	$I=\frac{\sqrt{x^2-9}}{9x}+c$

10	$\int \frac{dx}{x^2-2x+5}$	$I=\frac{1}{2}\tan^{-1}(\frac{x-1}{2})+c$
11	$\int \frac{dx}{4+x^2}$	$I=\frac{1}{2}\tan^{-1}(\frac{x}{2})+c$
12	$\int \frac{xdx}{4+x^2}$	$I=\frac{1}{2}\ln(x^2+4)+c$
13	$\int \frac{(x-1)dx}{\sqrt{2x-x^2}}$	$I=-\sqrt{2x-x^2}+c$
14	$\int \frac{xdx}{\sqrt{5+4x-x^2}}$	$I=2\sin^{-1}(\frac{x-2}{3})-\sqrt{5+4x-x^2}+c$
15	$\int \frac{dx}{\sqrt{4-(x-2)^2}}$	$I=\sin^{-1}(\frac{x-2}{2})+c$
16	$\int \frac{(1-x)dx}{\sqrt{8+2x-x^2}}$	$I=\sqrt{8+2x-x^2}+c$
17	$\int \frac{(x+1)dx}{\sqrt{4-x^2}}$	$I=\sin^{-1}(\frac{x}{2})-\sqrt{4-x^2}+c$
18	$\int \frac{xdx}{\sqrt{4x^2-4x+3}}$	$I=\frac{1}{4}\sinh^{-1}(\frac{2x-1}{\sqrt{2}})+\frac{1}{4}\sqrt{4x^2-4x+3}+c$
19	$\int \frac{dx}{\sqrt{2-5x^2}}$	$I=\frac{1}{\sqrt{5}}\sin^{-1}(\sqrt{\frac{5}{2}}x)+c$
20	$\int \frac{(2x+3)dx}{4x^2+4x+5}$	$I=\sinh^{-1}(\frac{2x+1}{2})+\frac{1}{2}\sqrt{4x^2+4x+5}+c$
21	$\int \frac{\sin\theta d\theta}{(2-\cos^2\theta)^{3/2}}$	$I=-\frac{\cos\theta}{2\sqrt{2-\cos^2\theta}}+c$
22	$\int \frac{dx}{(x^2-x+1)^{1/2}}$	$I=\sinh^{-1}(\frac{2x-1}{\sqrt{3}})+c$
23	$\int \frac{dx}{x\sqrt{x^2-a^2}}$	$I=\frac{1}{a}\tan^{-1}(\frac{\sqrt{x^2-a^2}}{a})+c$
24	$\int \frac{dx}{x\sqrt{3x^2+2x-1}}$	$I=\tan^{-1}(\frac{x-1}{\sqrt{3x^2+2x-1}})+c$

Exercise (7-3): Trigonometric substitutions method

Exe. (8.3) - page 452 Calculus-12

No.	Question	Answer
Using Trigonometric Substitutions		

Evaluate the integrals in Exercises 1-14.		
1	$\int \frac{dx}{\sqrt{49+x^2}}$	$I=\sinh^{-1}(\frac{x}{7})+c$
2	$\int \frac{3dx}{\sqrt{1+9x^2}}$	$I=\sinh^{-1}(3x)+c$
3	$\int_{-2}^{2} \frac{dx}{4+x^2}$	$I=\pi/4$
4	$\int_{0}^{2} \frac{dx}{8+2x^2}$	$I=\pi/16$
5	$\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$	$I=\pi/6$
6	$\int_{0}^{1/2\sqrt{2}} \frac{2dx}{\sqrt{1-4x^2}}$	$I=\pi/4$
7	$\int \sqrt{25-t^2}\,dt$	$I=\frac{25}{2}\sin^{-1}(\frac{t}{5})+\frac{t}{2}\sqrt{25-t^2}+c$
8	$\int \sqrt{1-9t^2}\,dt$	$I=\frac{1}{6}\sin^{-1}(3t)+\frac{t}{6}\sqrt{1-9t^2}+c$
9	$\int \frac{dx}{\sqrt{4x^2-49}}, \quad x>\frac{7}{2}$	$I=\frac{1}{2}\cosh^{-1}(\frac{7x}{2})+c$
10	$\int \frac{5dx}{\sqrt{25x^2-9}}, \quad x>\frac{3}{5}$	$I=\cosh^{-1}(\frac{5x}{3})+c$
11	$\int \frac{\sqrt{y^2-49}}{y}dy, \quad y>7$	$I=-7\tan^{-1}(\frac{\sqrt{y^2-49}}{7})+\sqrt{y^2-49}+c$
12	$\int \frac{\sqrt{y^2-25}}{y^3}dy, \quad y>5$	$I=\frac{1}{10}\tan^{-1}(\frac{\sqrt{y^2-25}}{5})-\frac{\sqrt{y^2-25}}{2y^2}+c$
13	$\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x>1$	$I=\frac{\sqrt{x^2-1}}{x}+c$
14	$\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x>1$	$I=\frac{1}{2}\tan^{-1}(\sqrt{x^2-1})+\frac{1}{2}\frac{\sqrt{x^2-1}}{x^2}+c$
Assorted Integrations Use any method to evaluate the integrals in Exercises 15-34. Most will require trigonometric substitutions, but some can be evaluated by other methods.		
15	$\int \frac{xdx}{\sqrt{9-x^2}}$	$I=-\sqrt{9-x^2}+c$

16	$\int \frac{x^2}{4+x^2}dx$	$I = x - 2\tan^{-1}(\frac{x}{2}) + c$
17	$\int \frac{x^3 dx}{\sqrt{x^2+4}}$	$I = \frac{1}{3}(x^2-8)\sqrt{x^2+4} + c$
18	$\int \frac{dx}{x^2\sqrt{x^2+1}}$	$I = -\frac{\sqrt{x^2+1}}{x} + c$
19	$\int \frac{8dw}{w^2\sqrt{4-w^2}}$	$I = -\frac{\sqrt{4-w^2}}{w} + c$
20	$\int \frac{\sqrt{9-w^2}}{w^2}dw$	$I = -\frac{\sqrt{9-w^2}}{w} + \sin^{-1}(\frac{x}{3}) + c$
21	$\int \frac{100}{36+25x^2}dx$	$I = -\frac{10}{3}\tan^{-1}(\frac{5x}{6}) + c$
22	$\int x\sqrt{x^2-4}dx$	$I = \frac{(x^2-4)^{3/2}}{3}$
23	$\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{(1-x^2)^{3/2}}$	$I = -\frac{4(\pi - 3^{3/2})}{3}$
24	$\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$	$I = \frac{1}{4\sqrt{3}}$
25	$\int \frac{dx}{(x^2-1)^{3/2}} \quad , \quad x>1$	$I = -\frac{x}{\sqrt{x^2-1}} + c$
26	$\int \frac{x^2 dx}{(x^2-1)^{5/2}} \quad , \quad x>1$	$I = -\frac{x^3}{3(x^2-1)^{3/2}} + c$
27	$\int \frac{(1-x^2)^{3/2}}{x^6}dx$	$I = -\frac{(1-x^2)^{5/2}}{5x^5} + c$
28	$\int \frac{(1-x^2)^{1/2}}{x^4}dx$	$I = -\frac{(1-x^2)^{3/2}}{3x^3} + c$
29	$\int \frac{8dx}{(4x^2+1)^2}$	$I = \frac{4x}{4x^2+1} + 2\tan^{-1}(2x) + c$
30	$\int \frac{6dt}{(9t^2+1)^2}$	$I = \frac{3x}{9x^2+1} + \tan^{-1}(3x) + c$
31	$\int \frac{x^3 dx}{x^2-1}$	$I = -\frac{x}{\sqrt{x^2-1}} + c$

		$I=\frac{1}{2}(x^2+\ln(x^2-1))+c$
32	$\int\frac{xdx}{25+4x^2}$	$I=\frac{1}{8}\ln(4x^2+25)+c$
33	$\int\frac{v^2dv}{(1-v^2)^{5/2}}$	$I=\frac{v^3}{3(1-v^2)^{3/2}}+c$
34	$\int\frac{(1-r^2)^{5/2}}{r^8}dr$	$I=-\frac{(1-r^2)^{7/2}}{7r^7}+c$
In Exercises 35-49, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.		
35	$\int_0^{\ln 4}\frac{e^t dt}{\sqrt{e^{2t}+9}}$	$I=\sinh^{-1}(\frac{4}{3})-\sinh^{-1}(\frac{1}{3})$
36	$\int_{\ln(3/4)}^{\ln(4/3)}\frac{e^t dt}{(1+e^{2t})^{3/2}}$	$I=1/5$
37	$\int_{1/12}^{1/4}\frac{2dt}{\sqrt{t}+4t\sqrt{t}}$	$I=\pi/6$
38	$\int_1^e\frac{dy}{y\sqrt{1+(\ln y)^2}}$	$I=\sinh^{-1}(1)$
39	$\int\frac{dx}{x\sqrt{x^2-1}}$	$I=\tan^{-1}(\sqrt{x^2-1})+c$
40	$\int\frac{dx}{1+x^2}$	$I=\tan^{-1}x+c$
41	$\int\frac{xdx}{\sqrt{x^2-1}}$	$I=\sqrt{x^2-1}+c$
42	$\int\frac{dx}{\sqrt{1-x^2}}$	$I=\sin^{-1}(x)+c$
43	$\int\frac{xdx}{\sqrt{1+x^4}}$	$I=\frac{1}{2}\sinh^{-1}(x^2)+c$
44	$\int\frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx$	$I=\sqrt{1-(\ln x)^2}-\tanh^{-1}(\sqrt{1-(\ln x)^2})$
45	$\int\sqrt{\frac{4-x}{x}}dx \quad ,(x=u^2)$	$I=x\sqrt{\frac{4-x}{x}}-4\tan^{-1}(\sqrt{\frac{4-x}{x}})+c$

46	$\int\sqrt{\frac{x}{1-x^3}}dx \quad , (u=x^{3/2})$	$I=\frac{2}{3}\sin^{-1}(x^{3/2})+c$
47	$\int\sqrt{x}\sqrt{1-x}dx$	$I=\frac{1}{4}[\sqrt{(1-x)x}(2x-1)-\sin^{-1}(\sqrt{1-x})]+c$
48	$\int\frac{\sqrt{x-2}}{\sqrt{x-1}}dx$	$I=\frac{\sqrt{x-1}(x-2)+\sqrt{2-x}\sin^{-1}(\sqrt{2-x})}{\sqrt{x-2}}+c$
49	$\int x^2\sqrt{x^2-4}dx$	$I=\frac{1}{4}\sqrt{x^2-4}(x^3-2x)-2\ln(\sqrt{x^2-4}+x)+c$
Initial Value Problems Solve the initial value problems in Exercises 50-53 for *y* as a function of *x*.		
50	$x\frac{dy}{dx}=\sqrt{x^2-4}, x\geq 2, y(2)=0$	$y=2\left[\frac{\sqrt{x^2-4}}{2}-\sec^{-1}\frac{x}{2}\right]$
51	$\sqrt{x^2-9}\frac{dy}{dx}=1, x>3, y(5)=\ln 3$	$y=\ln\left\|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right\|$
52	$(x^2+4)\frac{dy}{dx}=3, y(2)=0$	$y=\frac{3}{2}\tan^{-1}(\frac{x}{2})-\frac{3\pi}{8}$
53	$(x^2+1)^2\frac{dy}{dx}=\sqrt{x^2+1}, y(0)=1$	$y=\frac{x}{\sqrt{x^2+1}}+1$

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