جامعة تكريت – كلية التربية للبنات – قسم الرياضيات

المرحلة الثانية – المعادلات التفاضلية الاعتيادية

الفصل التمهيدي – طرائق التكامل

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عنوان المحاضرة :

الطريقة الثامنة عشر : تكامل الجذر ألتربيعي لدوال من الدرجة الثانية

الطريقة التاسعة عشر : تكامل الدوال النسبية المثلثية باستخدام دوال مثلثية أخرى

الطريقة العشرون : تكامل الدوال من النوع $\int \sec(x) \csc^2(x)$.

الطريقة الثامنة عشر Eighteenth method

تكامل الجذر ألتربيعي لدوال من الدرجة الثانية

Integral involving the square root of quadratic functions

يكون تكامل المقدار $\sqrt{ax^2+bx+c}$ بتحويل الدالة التربيعية الموجودة تحت الجذر إلى مربع كامل كما في المثال التالي

Example (76): Evaluate $I=\int \frac{dx}{\sqrt{2x-x^2}}$

$$2x-x^2=-x^2+2x=-(x^2-2x)=-(x^2-2x+1-1)=-(x^2-2x+1)+1$$

$$=-(x-1)^2+1=1-(x-1)^2$$

$$I=\int \frac{dx}{\sqrt{2x-x^2}}=\int \frac{dx}{\sqrt{1-(x-1)^2}}=\sin^{-1}(x-1)+c$$

Example (77): Evaluate $I=\int \frac{(x-1)dx}{\sqrt{2x-x^2}}$

$$2x-x^2=-x^2+2x=-(x^2-2x)=-(x^2-2x+1-1)=-(x^2-2x+1)+1$$

$$=-(x-1)^2+1=1-(x-1)^2$$

$$I=\int \frac{(x-1)dx}{\sqrt{2x-x^2}}=\int \frac{(x-1)dx}{\sqrt{1-(x-1)^2}}=\int \left[1-(x-1)^2\right]^{-\frac{1}{2}}(x-1)dx=\int \frac{1}{2}\left[1-(x-1)^2\right]^{-\frac{1}{2}}(2)(x-1)dx$$

$$I=\frac{1}{2}\frac{\left[1-(x-1)^2\right]^{\frac{1}{2}}}{1/2}+c=\left[1-(x-1)^2\right]^{\frac{1}{2}}+c$$

$$I=\sqrt{1-(x-1)^2}+c$$

الطريقة التاسعة عشر Nineteenth method

تكامل الدوال النسبية المثلثية باستخدام دوال مثلثية أخرى

اذا كانت الدالة المراد تكاملها نسـبية مثلثية (يصــعب تكاملها) فيمكن تحويلها الى دالة نسـبية جبرية بدلالة متغير اخر (فيسهل تكاملها) وذلك باستخدام الفرضيات التالية

let $z=\tan\frac{x}{2}$

$$\tan x=\frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}=\frac{2z}{1-z^2}$$

$$\cos x = 2\cos^2(\frac{x}{2}) - 1 = \frac{2}{\sec^2(\frac{x}{2})} - 1 = \frac{2}{1+\tan^2(\frac{x}{2})} - 1 = \frac{2}{1+z^2} - 1 = \frac{2-1-z^2}{1+z^2} = \frac{1-z^2}{1+z^2}$$

$$\sin x = 2\sin(\frac{x}{2})\cos(\frac{x}{2}) = 2\frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})}\cos^2(\frac{x}{2}) = 2\tan(\frac{x}{2})\frac{1}{\sec^2(\frac{x}{2})} = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{2z}{1+z^2}$$

$$z = \tan\frac{x}{2} \quad \Rightarrow \quad \frac{x}{2} = \tan^{-1} z \quad \Rightarrow \quad x = 2\tan^{-1} z \quad \Rightarrow \quad dx = 2\frac{1}{1+z^2}dz$$

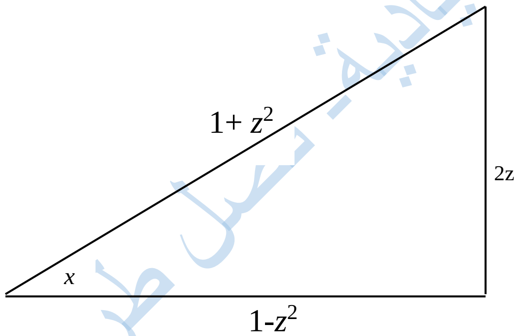
$$dx = \frac{2dz}{1+z^2}$$

كما يمكننا إيجاد العلاقات الأخرى باستخدام المثلث التالي.

$$\sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$dx = \frac{2dz}{1+z^2}$$



Example (78): Evaluate $I = \int \frac{dx}{2+\sin x}$

$$\sin x = \frac{2z}{1+z^2} \quad , \quad dx = \frac{2dz}{1+z^2}$$

$$I = \int \frac{dx}{2+\sin x} = \int \frac{\frac{2dz}{1+z^2}}{2+\frac{2z}{1+z^2}} = \int \frac{\frac{2dz}{1+z^2}}{\frac{2+2z^2+2z}{1+z^2}} = \int \frac{2dz}{2(z^2+z+1)} = \int \frac{dz}{z^2+z+1}$$

$$z^2+z+1 = z^2+z+\frac{1}{4}+\frac{3}{4} = (z+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2$$

$$I = \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \frac{2}{\sqrt{3}}\tan^{-1}[\frac{z+(1/2)}{(\sqrt{3}/2)}] + c$$

$$I = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{z+(1/2)}{(\sqrt{3}/2)}\right] + c = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\tan(x/2)+(1/2)}{(\sqrt{3}/2)}\right] + c$$

Example (79): Evaluate $I=\int \frac{dx}{1+\cos x}$

$$\cos x=\frac{1-z^2}{1+z^2} \quad , \quad dx=\frac{2dz}{1+z^2}$$

$$I=\int \frac{dx}{1+\cos x}=\int \frac{\frac{2dz}{1+z^2}}{1+\frac{1-z^2}{1+z^2}}=\int \frac{\frac{2dz}{1+z^2}}{\frac{1+z^2+1-z^2}{1+z^2}}=\int \frac{\frac{2dz}{1+z^2}}{\frac{2}{1+z^2}}=\int dz=z+c=\tan(\frac{x}{2})+c$$

Example (80): Evaluate $I=\int \frac{\sin xdx}{\cos^2 x+\cos x-2}$

$$\sin x=\frac{2z}{1+z^2} \quad , \quad \cos x=\frac{1-z^2}{1+z^2} \quad , \quad dx=\frac{2dz}{1+z^2}$$

$$I=\int \frac{\sin xdx}{\cos^2 x+\cos x-2}=\int \frac{\frac{2z}{1+z^2}\cdot\frac{2dz}{1+z^2}}{\left(\frac{1-z^2}{1+z^2}\right)^2+\left(\frac{1-z^2}{1+z^2}\right)-2}$$

$$=\int \frac{\frac{4zdz}{(1+z^2)^2}}{\frac{(1-z^2)^2+(1-z^2)(1+z^2)-2(1-z^2)^2}{(1+z^2)^2}}=\int \frac{4zdz}{1-2z^2+z^4+1-z^4-2-4z^2-2z^2}$$

$$=\int \frac{4zdz}{-6z^2-2z^4}=\int \frac{4zdz}{-2z^2(3+z^2)}=-2\int \frac{1}{z(3+z^2)}dz$$

$$\frac{1}{z(3+z^2)}=\frac{A}{z}+\frac{Bz+C}{3+z^2}=\frac{3A+Az^2+Bz^2+Cz}{z(3+z^2)}=\frac{(A+B)z^2+Cz+3A}{z(3+z^2)}$$

$$\left.\begin{aligned} A+B&=0\\ C&=0\\ 3A&=1 \end{aligned}\right\} \Rightarrow \begin{aligned} B&=-1/3\\ C&=0\\ A&=1/3 \end{aligned}$$

$$\frac{1}{z(3+z^2)}=\frac{1/3}{z}+\frac{-1/3z}{3+z^2}$$

$$I=-2\int \frac{1}{z(3+z^2)}dz=-2\int (\frac{1/3}{z}+\frac{-1/3z}{3+z^2})dz=\frac{-2}{3}\ln z+\frac{1}{3}\ln(3+z^2)+c$$

$$I=\frac{-2}{3}\ln(\tan(\frac{x}{2}))+\frac{1}{3}\ln(3+\tan^2(\frac{x}{2}))+c$$

Example (81): Evaluate $I=\int \frac{\cos xdx}{\cos^2 x+\cos x-2}$

$$\cos x=\frac{1-z^2}{1+z^2} \quad , \quad dx=\frac{2dz}{1+z^2}$$

$$I=\int\frac{\cos xdx}{\cos^2 x+\cos x-2}=\int\frac{\frac{1-z^2}{1+z^2}\cdot\frac{2dz}{1+z^2}}{\left(\frac{1-z^2}{1+z^2}\right)^2+\left(\frac{1-z^2}{1+z^2}\right)-2}$$

$$=\int\frac{\frac{2(1-z^2)dz}{(1+z^2)^2}}{\frac{(1-z^2)^2+(1-z^2)(1+z^2)-2(1-z^2)^2}{(1+z^2)^2}}=\int\frac{2(1-z^2)dz}{1-2z^2+z^4+1-z^4-2-4z^2-2z^2}$$

$$=\int\frac{2(1-z^2)dz}{-6z^2-2z^4}=\int\frac{2(1-z^2)dz}{-2z^2(3+z^2)}=\int\frac{z^2-1}{z^2(z^2+3)}dz$$

$$\frac{z^2-1}{z^2(z^2+3)}=\frac{A}{z}+\frac{B}{z^2}+\frac{Cz+D}{z^2+3}=\frac{Az(z^2+3)+B(z^2+3)+z^2(Cz+D)}{z^2(z^2+3)}$$

$$=\frac{Az^3+3Az+Bz^2+3B+Cz^3+Dz^2}{z^2(z^2+3)}=\frac{(A+C)z^3+(B+D)z^2+3Az+3B}{z^2(z^2+3)}$$

$$\left.\begin{array}{l}A+C=0\\ B+D=1\\ 3A=0\\ 3B=-1\end{array}\right\}\Rightarrow\begin{array}{l}C=0\\ D=4/3\\ A=0\\ B=-1/3\end{array}$$

$$\frac{z^2-1}{z^2(z^2+3)}=\frac{-1/3}{z^2}+\frac{4/3}{z^2+3}$$

$$I=\int\frac{z^2-1}{z^2(z^2+3)}dz=\int\frac{-1/3}{z^2}dz+\int\frac{4/3}{z^2+3}dz=\frac{-1}{3}\int z^{-2}dz+\frac{4}{3}\int\frac{dz}{z^2+3}=\frac{1}{3z}+\frac{4}{3}\int\frac{dz}{z^2+(\sqrt{3})^2}$$

$$=\frac{1}{3z}+\frac{4}{3}\cdot\frac{1}{\sqrt{3}}\tan^{-1}[\frac{z}{\sqrt{3}}]+c=\frac{1}{3\tan(x/2)}+\frac{4}{3\sqrt{3}}\tan^{-1}[\frac{\tan(x/2)}{\sqrt{3}}]+c$$

....

Exercise (8-1):

Exe. (37) page (311) – book al-samarrai

No.	Question	Answer
1	$\int\frac{\sin xdx}{1+\sin x}$	$I=x-\tan x+\sec x+c$
2	$\int\frac{3dx}{\sin x+\cos x}$	$I=-\frac{3}{\sqrt{2}}\tanh^{-1}\left(\sqrt{2}\cos x\right)-\frac{3}{\sqrt{2}}\tanh^{-1}\left(\sqrt{2}\sin x\right)+c$
3	$\int\frac{dx}{1+\sin x-\cos x}$	$I=\ln(\tan\frac{x}{2})+\ln(\tan\frac{x}{2}+1)+c$
4	$\int\frac{\sin x\cos x}{1-\cos x}dx$	$I=\cos x+\ln(\cos x-1)+c$
5	$\int\frac{dx}{5+4\sin x}$	$I=\frac{2}{3}\tan^{-1}(\frac{5}{3}\tan(\frac{x}{2})+\frac{4}{3})+c$

6	$\int \frac{dx}{3-2\cos x}$	$I=\frac{2}{\sqrt{5}}\tan^{-1}(\sqrt{5}\tan(\frac{x}{2}))+c$
7	$\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}}$	$I=2\sqrt{x}+4\sqrt[4]{x}+4\ln(1-\sqrt[4]{x})+c$
8	$\int \frac{dx}{x\sqrt{1-x}}$	$I=-2\tanh^{-1}(\sqrt{1-x})+c$
9	$\int x^5\sqrt{1-x^2}\,dx$	$I=\frac{-1}{105}(1-x^2)^{3/2}(15x^4+12x^2+8)+c$
10	$\int \frac{\sqrt{x}dx}{1+x}$	$I=2\sqrt{x}-2\tan^{-1}(\sqrt{x})+c$
11	$\int \frac{dx}{3+\sqrt{x+2}}$	$I=2\sqrt{x+2}-6\ln(\sqrt{x+2}+3)+c$
12	$\int_{\pi/2}^{\pi} \frac{dx}{1-\cos x}$	1
13	$\int_{0}^{\pi/2} \frac{dx}{2+\cos x}$	$\pi/3\sqrt{3}$
14	$\int_{0}^{3} \frac{xdx}{\sqrt{1+x}}$	$8/3$
15	$\int_{1}^{64} \frac{dx}{\sqrt[3]{x}+2\sqrt{x}}$	$\frac{1}{8}(44-3\log(\frac{5}{3}))$

Exercise (8-2):

Exe. (6-12) page (414) blue book

No.	Question	Answer
1	$\int \frac{dx}{2+\cos x}$	$I=\frac{2}{\sqrt{3}}\tan^{-1}(\frac{1}{\sqrt{3}}\tan(\frac{x}{2}))+c$
2	$\int \frac{dx}{13+5\cos x}$	$I=\frac{1}{6}\tan^{-1}(\frac{2}{3}\tan(\frac{x}{2}))+c$
3	$\int \frac{dx}{\sin x-\cos x}$	$I=-\sqrt{2}\left(\frac{1}{2}\ln\left\|\frac{\tan(x/2)+1}{\sqrt{2}}+1\right\|-\frac{1}{2}\ln\left\|\frac{\tan(x/2)+1}{\sqrt{2}}-1\right\|\right)+c$
4	$\int \sqrt{1-\sin x}\,dx$	$I=2\sqrt{\sin x+1}+c$
5	$\int \frac{\cot xdx}{1-\cos x}$	$I=\frac{\ln(\cos x+1)-\ln(1-\cos x)}{4}+\frac{1}{2\cos x-2}+c$
6	$\int \frac{\cos xdx}{2-\cos x}$	$I=-x+\frac{4}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan(\frac{x}{2}))+c$
7	$\int \frac{dx}{5+4\sin x}$	$I=\frac{2}{3}\tan^{-1}(\frac{5}{3}\tan(\frac{x}{2})+\frac{4}{3})+c$

8	$\int \frac{dx}{\sin x + \tan x}$	$I = \frac{-1}{4}\tan^2(\frac{x}{2}) + \frac{1}{2}\ln\left\|\tan(\frac{x}{2})\right\| + c$
9	$\int \frac{dx}{1+\sin x + \cos x}$	$I = \ln(\|\tan(x/2)+1\|) + c$

Exercise (8-3): Exe. (8.5) - page 467 Calculus-12

No.	Question	Answer
Using Integral Tables: use the table of integrals at the back of the book to evaluate the integrals in Exercises 1-26. or evaluate by any suitable method		
1	$\int \frac{dx}{x\sqrt{x-3}}$	$I = \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{x-3}{3}} + c$
2	$\int \frac{dx}{x\sqrt{x+4}}$	$I = \frac{1}{2}\ln\left\|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right\| + c$
3	$\int \frac{xdx}{\sqrt{x-2}}$	$I = \sqrt{x-2}\left[\frac{2(x-2)}{3}+4\right] + c$
4	$\int \frac{xdx}{(2x+3)^{3/2}}$	$I = \frac{x+3}{\sqrt{2x+3}} + c$
5	$\int x\sqrt{2x-3}dx$	$I = \frac{(2x+3)^{3/2}(x+1)}{5} + c$
6	$\int x(7x+5)^{3/2}dx$	$I = \frac{(7x+5)^{5/2}}{49}\cdot\frac{(14x-4)}{7} + c$
7	$\int \frac{\sqrt{9-4x}}{x^2}dx$	$I = \frac{-\sqrt{9-4x}}{x} - \frac{2}{3}\ln\left\|\frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3}\right\| + c$
8	$\int \frac{dx}{x^2\sqrt{4x-9}}$	$I = \frac{\sqrt{4x-9}}{9x} + \frac{4}{27}\tan^{-1}\sqrt{\frac{4x-9}{9}} + c$
9	$\int x\sqrt{4x-x^2}dx$	$I = \frac{(x+2)(x-3)\sqrt{4x-x}}{3} + 4\sin^{-1}(\frac{x-2}{2}) + c$
10	$\int \frac{\sqrt{x-x^2}}{x}dx$	$I = \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + c$
11	$\int \frac{dx}{x\sqrt{7+x^2}}$	$I = \frac{-1}{\sqrt{7}}\ln\left\|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right\| + c$
12	$\int \frac{dx}{x\sqrt{7-x^2}}$	$I = \frac{-1}{\sqrt{7}}\ln\left\|\frac{\sqrt{7}+\sqrt{7-x^2}}{x}\right\| + c$

13	$\int \frac{\sqrt{4-x^2}}{x}dx$	$I=\sqrt{4-x^2}-2\ln\left\lvert\frac{2+\sqrt{4-x^2}}{x}\right\rvert+c$
14	$\int \frac{\sqrt{x^2-4}}{x}dx$	$I=\sqrt{x^2-4}-2\sec^{-1}\left\lvert\frac{x}{2}\right\rvert+c$
15	$\int e^{2t}\cos 3t dt$	$I=\frac{e^{2t}}{13}(2\cos 3t+3\sin 3t)+c$
16	$\int e^{-3t}\sin 4t dt$	$I=\frac{e^{-3t}}{25}(-3\sin 4t-4\cos 4t+)+c$
17	$\int x\cos^{-1}x dx$	$I=\frac{x^2}{2}\cos^{-1}x+\frac{1}{4}\sin^{-1}x-\frac{1}{4}x\sqrt{1-x^2}+c$
18	$\int x\tan^{-1}x dx$	$I=\frac{1}{2}((x^2+1)\tan^{-1}x+x)+c$
19	$\int x^2\tan^{-1}x dx$	$I=\frac{x^2}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\ln(1+x^2)+c$
20	$\int \frac{\tan^{-1}x}{x^2}dx$	$I=\frac{-1}{x}\tan^{-1}x+\ln\lvert x\rvert-\frac{1}{2}\ln(1+x^2)+c$
21	$\int \sin 3x\cos 2x dx$	$I=\frac{-\cos 5x}{10}-\frac{\cos x}{2}+c$
22	$\int \sin 2x\cos 3x dx$	$I=\frac{-\cos 5x}{10}+\frac{\cos x}{2}+c$
23	$\int 8\sin 4t\sin\frac{t}{2}dt$	$I=\frac{8}{7}\sin\frac{7t}{2}-\frac{8}{9}\sin\frac{9t}{2}+c$
24	$\int \sin\frac{t}{3}\sin\frac{t}{6}dt$	$I=3\sin\frac{t}{6}-\sin\frac{t}{2}+c$
25	$\int \cos\frac{\theta}{3}\cos\frac{\theta}{4}d\theta$	$I=6\sin\frac{\theta}{12}+\frac{6}{7}\sin\frac{7\theta}{12}+c$
26	$\int \cos\frac{\theta}{2}\cos 7\theta d\theta$	$I=\frac{1}{13}\sin\frac{13\theta}{2}+\frac{1}{15}\sin\frac{15\theta}{2}+c$
Substitution and Integral Tables In Exercises 27-40, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.		
27	$\int \frac{x^2+x+1}{(x^2+1)^2}dx$	$I=\frac{1}{2}\ln(x^2+1)+\frac{x}{2(1+x^2)}+\frac{1}{2}\tan^{-1}x+c$
28	$\int \frac{x^2+6x}{(x^2+3)^2}dx$	$I=\frac{1}{2\sqrt{3}}\tan^{-1}(\frac{x}{\sqrt{3}})-\frac{3}{x^2+3}-\frac{x}{2(x^2+3)}+c$
29	$\int \sin^{-1}\sqrt{x}dx$	$I=(x-\frac{1}{2})\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+c$

30	$\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}dx$	$I=2(\sqrt{x}\cos^{-1}\sqrt{x}-\sqrt{1-x})+c$
31	$\int \frac{\sqrt{x}}{\sqrt{1-x}}dx$	$I=\sin^{-1}\sqrt{x}-\sqrt{x-x^2}+c$
32	$\int \frac{\sqrt{2-x}}{\sqrt{x}}dx$	$I=\sqrt{2x-x^2}+2\sin^{-1}\sqrt{\frac{x}{2}}+c$
33	$\int \cot t\sqrt{1-(\sin t)^2}\cdot d$	$I=\sqrt{1-\sin^2 t}-\ln\left\|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right\|+c$
34	$\int \frac{dt}{\tan t\sqrt{4-(\sin t)^2}}$	$I=-\frac{1}{2}\ln\left\|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right\|+c$
35	$\int \frac{dy}{y\sqrt{3+(\ln y)^2}}$	$I=\sin^{-1}(\frac{\ln x}{\sqrt{3}})+c$
36	$\int \tan^{-1}\sqrt{y}dy$	$I=y\tan^{-1}\sqrt{y}+\tan^{-1}\sqrt{y}-\sqrt{y}+c$
37	$\int \frac{dx}{\sqrt{x^2+2x+5}}$, **hint: complete the square**	$I=\sinh^{-1}(\frac{2x+2}{4})+c$
38	$\int \frac{x^2dx}{\sqrt{x^2-4x+5}}$	$I=\frac{1}{2}\left(7\sinh^{-1}(\frac{2x-4}{2})+(x+6)\sqrt{x^2-4x+5}\right)+c$
39	$\int \sqrt{5-4x-x^2}dx$	$I=\frac{x+2}{2}\sqrt{5-4x-x^2}+\frac{9}{2}\sin^{-1}(\frac{x+2}{3})+c$
40	$\int x^2\sqrt{2x-x^2}dx$	$I=\frac{1}{24}\sqrt{2x-x^2}(6x^3-2x^2-5x-15)-\frac{5}{4}\sin^{-1}\sqrt{\frac{2-x}{2}}+c$
Using Reduction Formulas Use reduction formulas to evaluate the integrals in Exercises 41-50.		
41	$\int \sin^5 2xdx$	$I=-\frac{\sin^4 2x\cos 2x}{10}-\frac{2\sin^2 2x\cos 2x}{15}-\frac{4\cos 2x}{15}+c$
42	$\int 8\cos^4 2\pi tdt$	$I=\frac{\cos^3 2\pi t\sin 2\pi t}{\pi}+\frac{3\cos 2\pi t\sin 2\pi t}{2\pi}+3t+c$
43	$\int \sin^2 2\theta\cos^3 2\theta d\theta$	$I=\frac{\sin^3 2\theta\cos^2 2\theta}{10}+\frac{\sin^3 2\theta}{15}+c$
44	$\int 2\sin^2 t\sec^4 tdt$	$I=\frac{2}{3}\tan^3 t+c+c$
45	$\int 4\tan^3 2xdx$	$I=\tan^2 2x-2\ln\|\sec 2x\|+c$
46	$\int 8\cot^4 tdt$	$I=8(\frac{-1}{3}\cot^3 t+\cot t+t)+c$

47	$\int 2\sec^3 \pi x dx$	$I = \dfrac{\sec \pi x \tan \pi x + \ln\lvert \sec \pi x + \tan \pi x\rvert}{\pi} + c$
48	$\int 3\sec^4 3x dx$	$I = \dfrac{\sec^2 3x \tan 3x + 2\tan 3x}{3} + c$
49	$\int \csc^5 x dx$	$I = \dfrac{-2\csc^3 x \cot x - 3\csc x \cot x - 3\ln\lvert \csc x \cot x\rvert}{8} + c$
50	$\int 16x^3 (\ln x)^2 dx$	$I = 4x^4(\ln x)^2 - 2x^4(\ln x) + \dfrac{x^4}{2} + c$

Evaluate the integrals in Exercises 51-56 by making a substitution (possibly trigonometric) and then applying a reduction formula. or evaluate by any suitable method

51	$\int e^t \sec^3(e^t - 1)dt$	$I = \dfrac{\sec(e^t - 1)\tan(e^t - 1) + \ln\lvert \sec(e^t - 1) + \tan(e^t - 1)\rvert}{2} + c$
52	$\int \dfrac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}} d\theta$	$I = -\csc\sqrt{\theta}\cot\sqrt{\theta} - \ln\lvert \csc\sqrt{\theta} + \cot\sqrt{\theta}\rvert + c$
53	$\int_0^1 2\sqrt{x^2+1}dx$	$\sqrt{2} + \ln(\sqrt{2}+1)$
54	$\int_0^{\sqrt{3}/2} \dfrac{dy}{(1-y^2)^{5/2}}$	$2\sqrt{3}$
55	$\int_1^2 \dfrac{(r^2-1)^{3/2}}{r} dy$	$\pi / 3$
56	$\int_0^{1/\sqrt{3}} \dfrac{dt}{(t^2+1)^{7/2}}$	$203/480$

Twentieth method   الطريقة العشرون

تكامل الدوال من النوع $\int \sec(x) \csc^2(x)$

Example (82): Evaluate $I = \int \sec x (\csc x)^2 dx$

$$I = \int \sec x (1 + \cot^2 x) dx$$

$$= \int \sec x dx + \int \sec x \cot^2 x dx$$

$$= \int \sec x dx + \int (\sec x \cot x) \cot x dx$$

$$= \int \sec x dx + \int \csc x . \cot x dx$$

$$= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx + \int \csc x . \cot x dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx + \int \csc x . \cot x dx$$

$$= \ln|\sec x + \tan x| - \csc x + c$$

Exercise (9-1): by clever rearrangement method evaluate

No.	Question	Answer
1	$\int x(x-1)^{10} dx$	
2	$\int x\sqrt{4-x} dx$	
3	$\int (x+1)^2(1-x)^5 dx$	
4	$\int (x+5)(x-5)^{1/3} dx$	
5	$\int x^3\sqrt{x^2+1} dx$	
6	$\int 3x^5\sqrt{x^3+1} dx$	

Exercise (10-1): By any method evaluate

Exe. (8.5) - page 467. Calculus-12

No.	Question	Answer
1	$\int \frac{dx}{x\sqrt{x-2}}$	$I = \sqrt{2}\tan^{-1}\left(\frac{\sqrt{x-2}}{\sqrt{2}}\right) + c$
2	$\int \frac{dx}{x\sqrt{x+9}}$	$I = -\frac{2}{3}\tanh^{-1}(\frac{\sqrt{x+9}}{3}) + c$
3	$\int \frac{xdx}{\sqrt{2-x}}$	$I = -\frac{2}{3}\sqrt{2-x}(x-4) + c$
4	$\int \frac{xdx}{(2x-3)^{5/2}}$	$I = \frac{1-x}{(2x-3)^{3/2}} + c$
5	$\int x\sqrt{3-2x} dx$	$I = \frac{-1}{5}(x+1)(3-2x)^{3/2} + c$
6	$\int x(5-x)^{2/3}$	$I = \frac{-3}{8}(x+3)(5-x)^{5/3} + c$
7	$\int \frac{\sqrt{5-x}}{x} dx$	$I = 2(\sqrt{5-x} - \sqrt{5}\tanh^{-1}(\sqrt{\frac{5-x}{5}})) + c$
8	$\int \frac{dx}{x^2\sqrt{4x-x^2}}$	$I = -\frac{\sqrt{-(x-4)x} \cdot (x+2)}{12x^2} + c$
9	$\int \sqrt{x^2-100}\, dx$	$I = \frac{x}{2}\sqrt{x^2-100} - 50\log(\sqrt{x^2-100} + x) + c$
10	$\int \frac{\sqrt{x^2-1}}{x} dx$	$I = \sqrt{x^2-1} - \tan^{-1}(\sqrt{x^2-1}) + c$
11	$\int \frac{dx}{\sqrt{x^2+7}}$	$I = \sinh^{-1}(\frac{x}{\sqrt{7}}) + c$

12	$\int \frac{dx}{\sqrt{7-x^2}}$	$I=\sin^{-1}(\frac{x}{\sqrt{7}})+c$
13	$\int \frac{\sqrt{4-x}}{x}dx$	$I=2\sqrt{4-x}-4\tanh^{-1}(\frac{\sqrt{4-x}}{2})+c$
14	$\int \frac{\sqrt{x-4}}{x}dx$	$I=2\sqrt{x-4}-4\tan^{-1}(\frac{\sqrt{x-4}}{2})+c$
15	$\int e^{3x}\cos 5x dx$	$I=\frac{e^{3x}}{34}(5\sin 5x+3\cos 5x)+c$
16	$\int e^{-3x}\cos 5x dx$	$I=\frac{e^{-3x}}{34}(5\sin 5x-3\cos 5x)+c$
17	$\int x\cos^{-1}(3x)dx$	$I=\frac{1}{36}\left(-3x\sqrt{1-9x^2}+18x^2\cos^{-1}(3x)+\sin^{-1}(3x)\right)+c$
18	$\int x\tan^{-1}(\frac{x}{2})dx$	$I=\frac{1}{2}(x^2+4)\tan^{-1}(\frac{x}{2})-x+c$
19	$\int \sqrt{x}\tan^{-1}\sqrt{x}dx$	$I=\frac{1}{3}\left(2x^{3/2}\tan^{-1}\sqrt{x}-x+\log(x+1)\right)+c$
20	$\int x\tan^{-1}\sqrt{x}dx$	$I=\frac{1}{6}\left((3x^2-3)\tan^{-1}\sqrt{x}-(x-3)\sqrt{x}\right)+c$
21	$\int \sin(-3x)\cos 2x dx$	$I=\frac{1}{10}(5\cos x+\cos 5x)+c$
22	$\int \sin(3x)\cos(-2x)dx$	$I=\frac{-1}{10}(5\cos x+\cos 5x)+c$
23	$\int \sin(6x)\sin(x/2)dx$	$I=\frac{1}{11}\sin(\frac{11x}{2})-\frac{1}{13}\sin(\frac{13x}{2})+c$
24	$\int \sin(x/4)\sin(x/8)dx$	$I=\frac{16}{3}\sin^3(\frac{x}{8})+c$
25	$\int \cos(x/4)\cos(x/8)dx$	$I=4\sin(x/8)+(4/3)\sin(3x/8)+c$
26	$\int \cos(x/4)\cos(-x/4)dx$	$I=(x/2)+\sin(x/2)+c$
27	$\int \frac{x^3+2x+1}{(x^2+1)^2}dx$	$I=\frac{1}{2}\left(\frac{x-1}{x^2+1}+\log(x^2+1)+\tan^{-1}x\right)+c$
28	$\int \frac{x^2+2x+1}{(x^2+1)^2}dx$	$I=\tan^{-1}x-\frac{1}{x^2+1}+c$
29	$\int \sqrt{x}\sin^{-1}\sqrt{x}dx$	$I=\frac{2}{9}\left(3x^{3/2}\sin^{-1}\sqrt{x}+\sqrt{1-x}(x+2)\right)+c$
30	$\int \frac{\sin^{-1}\sqrt{x}}{\sqrt{x}}dx$	$I=2\sqrt{1-x}+2\sqrt{x}\sin^{-1}\sqrt{x}+c$
31	$\int \frac{\sqrt{1-x}}{\sqrt{x}}dx$	$I=2\sqrt{(1-x)x}+\sin^{-1}\sqrt{x}+c$
32	$\int \frac{\sqrt{3-x}}{\sqrt{x}}dx$	$I=\sqrt{(3-x)x}+3\sin^{-1}\sqrt{\frac{x}{3}}+c$
33	$\int \tan x\sqrt{1-\cos^2 x}dx$	$I=-\sin x+\ln\left|\sec x+\tan x\right|+c$
34	$\int \frac{dx}{\tan x\sqrt{4+\sin^2 x}}$	$I=-\frac{1}{2}\ln\left|\sqrt{4\csc^2 x+1}+2\csc x\right|+c$

35	$\int \frac{dx}{x\sqrt{3-(\ln x)^2}}$	$I=\sin^{-1}\left(\frac{\ln x}{\sqrt{3}}\right)+c$
36	$\int \tan^{-1}\sqrt{x+1}dx$	$I=(x+2)\tan^{-1}\sqrt{x+1}-\sqrt{x+1}+c$
37	$\int \frac{dx}{\sqrt{x^2-2x+5}}$	$I=\sinh^{-1}\left(\frac{2x-2}{4}\right)+c$
38	$\int \frac{xdx}{\sqrt{x^2-4x+5}}$	$I=2\ln\left\|x+\sqrt{(x-2)^2+1}-2\right\|+\sqrt{x^2-4x+5}+c$
39	$\int x^3\sqrt{x^2-4}dx$	$I=\frac{1}{15}(x^2-4)^{3/2}(3x^2+8)+c$
40	$\int x^5\sqrt{x^3-4}dx$	$I=\frac{1}{45}(x^3-4)^{3/2}(6x^3+16)+c$
41	$\int \sin^5(x/2)dx$	$I=-\frac{2\cos^5(x/2)}{5}+\frac{4\cos^3(x/2)}{3}-2\cos(x/2)+c$
42	$\int \cos^5(x/3)dx$	$I=\frac{3\sin^5(x/3)}{5}-2\sin^3(x/3)+3\sin(x/3)+c$
43	$\int \sin^2(x/2)\cos^3(x/2)dx$	$I=\frac{2\sin^3(x/2)}{3}-\frac{2\sin^5(x/2)}{5}+c$
44	$\int \sin^2(x/2)\sec^4(x/2)dx$	$I=\frac{2}{3}\tan^3(x/2)+c$
45	$\int \tan^3(x/5)dx$	$I=\frac{5}{2}\tan^2(x/5)+5\log\left\|\cos(x/5)\right\|+c$
46	$\int \cot^4(2x)dx$	$I=x-\frac{1}{6}\cot^3(2x)+\frac{1}{2}\cot(2x)+c$
47	$\int \sec^3(7x)dx$	$I=\frac{1}{14}\left(\ln\left\|\tan 7x+\sec 7x\right\|+\sec 7x\tan 7x\right)+c$
48	$\int \sec^4(7x)dx$	$I=\frac{1}{21}\tan^3(7x)+\frac{1}{7}\tan(7x)+c$
49	$\int \csc(-5x)dx$	$I=\frac{1}{5}\ln\left\|\csc 5x+\cot 5x\right\|+c$
50	$\int x^2(\ln x)^2dx$	$I=\frac{1}{27}x^3[9\ln^2(x)-6\ln(x)+2]+c$
51	$\int e^{2x}(\csc e^{2x})^4dx$	$I=-\frac{\cot e^{2x}(\cot^2 e^{2x}+3)}{6}+c$
52	$\int \frac{\sec^3\sqrt{x}}{\sqrt{x}}dx$	$I=\sec\sqrt{x}\tan\sqrt{x}+\left(\ln\left\|\tan\sqrt{x}+\sec\sqrt{x}\right\|\right)+c$
53	$\int_{-1}^{1}\sqrt{1-x^2}dx$	$\pi/2$
54	$\int_{0}^{1/2}\frac{1}{(1-x^2)^{3/2}}dx$	$1/\sqrt{3}$
55	$\int_{1}^{2}(x^2-1)^{3/2}dx$	$\pi/3$
56	$\int_{-1}^{1}\frac{1}{(1+x^2)^{3/2}}dx$	$\sqrt{2}$

Exercise (10-2): By any method evaluate Exe. (38) page (311) – book al-samarrai

No.	Question	Answer
1	$\int \frac{(x+2)dx}{\sqrt{x^2+2x-4}}$	$I=\ln\left\|\sqrt{(x+1)^2-5}+x+1\right\|+\sqrt{x^2+2x-4}+c$
2	$\int \frac{\cos x dx}{\sqrt{1+\sin x}}$	$I=2\sqrt{1+\sin x}+c$
3	$\int x^2\cos\frac{x}{2}dx$	$I=2x^2\sin\frac{x}{2}+8x\cos\frac{x}{2}-16\sin\frac{x}{2}+c$
4	$\int \cos(\ln x)\frac{1}{x}dx$	$I=\sin(\ln x)+c$
5	$\int \frac{(3x-7)dx}{(x-1)(x-2)(x-3)}$	$I=-2\ln\|x-1\|+\ln\|x-2\|+\ln\|x-3\|+c$
6	$\int \frac{\sqrt{25-x^2}}{x}dx$	$I=\sqrt{25-x^2}-5\tanh^{-1}(\frac{\sqrt{25-x^2}}{5})+c$
7	$\int \frac{dy}{121+y^2}$	$I=\frac{1}{11}\tan^{-1}(\frac{y}{11})+c$
8	$\int \frac{3xdx}{\sqrt[3]{x^2+3}}$	$I=\frac{9}{4}(x^2+3)^{2/3}+c$
9	$\int \frac{x^2dx}{1-x^6}$	$I=\frac{1}{3}\tan^{-1}(x^2)+c$
10	$\int \frac{x^3dx}{1-x^2}$	$I=-\frac{x^2}{2}-\frac{1}{2}\ln\|x^2-1\|+c$
11	$\int \frac{dx}{3-e^{5x}}$	$I=\frac{x}{3}-\frac{1}{15}\ln\|e^{5x}-3\|+c$
12	$\int \frac{dx}{1+\cos^2 x}$	$I=\frac{1}{\sqrt{2}}\tan^{-1}(\tan\frac{x}{\sqrt{2}})+c$
13	$\int \frac{\cos xdx}{1+\cos^2 x}$	$I=\frac{1}{\sqrt{2}}\tanh^{-1}(\sin\frac{x}{\sqrt{2}})+c$
14	$\int \frac{dx}{e^{5x}+e^{-5x}}$	$I=\frac{1}{5}\tan^{-1}(e^{5x})+c$
15	$\int \frac{\cot xdx}{\ln(\sin x)}$	$I=\ln(\ln(\sin x))+c$
16	$\int \frac{dx}{\sqrt{1+e^x}}$	$I=\ln(\sqrt{1+e^x}-1)-\ln(\sqrt{1+e^x}+1)+c$
17	$\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$	$I=2\ln(\sqrt{x}+1)+c$
18	$\int x\sqrt{1-x}dx$	$I=\frac{2}{5}(1-x)^{5/3}-\frac{2}{3}(1-x)^{3/2}+c$
19	$\int \frac{e^{2x}-1}{e^{2x}+3}dx$	$I=\frac{-x}{3}+\frac{2}{3}\ln\|e^{2x}+3\|+c$

20	$\int \frac{dx}{2+\sin x}$	$I = \frac{2}{\sqrt{3}} \tan^{-1}(\frac{2\tan(x/2)+1}{\sqrt{3}}) + c$
21	$\int e^{\tan 2x} \sec^2 2x dx$	$I = \frac{1}{2} e^{\tan 2x} + c$
22	$\int \sin^3 2x \cos^3 2x dx$	$I = -\frac{1}{12}\sin^6(2x) - \frac{1}{8}\sin^4(2x) + c$
23	$\int \frac{dx}{x(\ln x + 2)}$	$I = \ln\lvert \ln x + 2 \rvert + c$
24	$\int \tan^6 x dx$	$I = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + c$
25	$\int (\ln x)^2 dx$	$I = x(\ln x)^2 - 2x\ln x + 2x + c$
26	$\int \frac{\sec x \tan x dx}{7+4\sec x}$	$I = \frac{1}{4}\ln(4\sec x + 7) + c$
27	$\int \frac{\sin x dx}{\cos^2 x - 5\cos x + 4}$	$I = \frac{1}{3}\ln(\cos x - 1) - \frac{1}{3}\ln(\cos x - 4) + c$
28	$\int \frac{\cot x dx}{1+\sin^2 x}$	$I = \frac{-1}{2}\ln(\cot^2 x + 2) + c$
29	$\int x\sec^2 x dx$	$I = x\tan x + \ln(\cos x) + c$
30	$\int \ln(x+\sqrt{1+x^2})dx$	$I = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + c$
31	$\int \frac{x^2 dx}{\sqrt{x^2-81}}$	$I = \frac{81}{2}\ln(\sqrt{x^2-81}+x) + \frac{1}{2}x\sqrt{x^2-81} + c$
32	$\int \frac{dx}{\sin x - \cos x - 1}$	$I = \ln\lvert \tan(x/2) - 1 \rvert + c$
33	$\int \frac{dx}{\sqrt{x^2 - 4x + 13}}$	$I = \sinh^{-1}(\frac{2x-4}{6}) + c$
34	$\int \tan^4 x \sec^4 x dx$	$I = \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + c$
35	$\int \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx$	$I = \frac{2}{3}\arctan^{3-2} x + c$
36	$\int \frac{\sin x dx}{1+\sin^2 x}$	$I = \frac{\ln(\sqrt{2}-\cos x) - \ln(\sqrt{2}+\cos x)}{2^{3/2}} + c$
37	$\int \sin\sqrt{2x} dx$	$I = \sin\sqrt{2x} - \sqrt{2x}\cos\sqrt{2x} + c$
38	$\int \frac{\sqrt{t} dt}{t^3+9}$	$I = \frac{2}{9}\arctan(\frac{x^{3/2}}{3}) + c$
39	$\int \frac{\ln x}{x} dx$	$I = \frac{1}{2}(\ln x)^2 + c$
40	$\int x^5 e^{x^3} dx$	$I = \frac{(x^3-1)e^{x^3}}{3} + c$

41	$\int \frac{x\,dx}{1+\sqrt{x}}$	$I = \frac{2\sqrt{x}(x+3)}{3} - x - 2\ln(\sqrt{x}+1) + c$
42	$\int \frac{x^3 dx}{\sqrt{9-x^2}}$	$I = -\frac{\sqrt{9-x^2}(x^2+18)}{3} + c$
43	$\int \frac{\cot^3 x}{\csc x} dx$	$I = -\sin x - \csc x + c$
44	$\int \tan^{3/2} x \sec^4 x\,dx$	$I = \frac{2}{5}\tan^{5/2} x + \frac{2}{9}\tan^{9/2} x + c$
45	$\int x \cdot \sqrt[3]{a^2x^2 + b^2}\,dx$	$I = \frac{3(a^2x^2+b)^{4/3}}{8a^2} + c$
46	$\int \frac{dx}{(x^2-4x+5)^2}$	$I = \frac{1}{2}[\frac{x-2}{x^2-4x+5} - \tan^{-1}(2-x)] + c$
47	$\int \frac{dx}{e^{2x} - 3e^x}$	$I = \frac{-x}{9} + \frac{1}{3e^x} + \frac{1}{9}\ln(e^x - 3) + c$
48	$\int \ln(x^2+1)dx$	$I = -2x + x\ln(x^2+1) + 2\tan^{-1}(x) + c$
49	$\int \ln(x^2+x)dx$	$I = -2x + \ln(x+1) + x\ln(x^2+x) + c$
50	$\int \frac{x^3+1}{x^3-x} dx$	$I = x - \ln x + \ln(x-1) + c$
51	$\int x\sin^2 x\,dx$	$I = \frac{x^2}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c\,2$
52	$\int (x+1)^2 e^x dx$	$I = e^x(x^2+1) + c$
53	$\int \frac{dt}{\sec^2 t + \tan^2 t}$	$I = \sqrt{2}\tan^{-1}(\sqrt{2}\tan t) + c$
54	$\int \frac{\sec^2 x\,dx}{\sqrt{4-\sec^2 x}}$	$I = \sin^{-1}(\frac{\tan x}{\sqrt{3}}) + c$
55	$\int \frac{dx}{\sin^3 x}$	$I = -\frac{\cot x \csc x}{2} - \frac{\ln(\cot x + \csc x)}{2} + c$
56	$\int x\ln\sqrt[3]{3x+1}\,dx$	$I = -\frac{(18x^2-2)\ln(3x+1) - 9x^2 + 6x}{108} + c$
57	$\int \sin^{-1}\sqrt{x}\,dx$	$I = \frac{\sqrt{x(1-x)} + (2x-1)\sin^{-1}(\sqrt{x})}{2} + c$
58	$\int \frac{\sin x\,dx}{\sin x + \cos x}$	$I = \frac{x - \ln(\sin x + \cos x)}{2} + c$
59	$\int \frac{\sin x\,dx}{\sin x - \cos x}$	$I = \frac{\ln(\sin x - \cos x) + x}{2} + c$
60	$\int \frac{e^{2x} dx}{1+e^x}$	$I = e^x - \ln(e^x+1) + c$