جامعة تكريت – كلية التربية للبنات – قسم الرياضيات

المرحلة الثانية – المعادلات التفاضلية الاعتيادية

الفصل التمهيدي – طرائق التكامل

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عنوان المحاضرة :

الطريقة الرابعة : ابدال المتغيرات

الطريقة الرابعة Fourth Method

تســـتخدم هذه الطريقة في الدوال التي تحتوي على جذور وفكرة هذه الطريقة هي ابدال المتغيرات وذلك بفرض علاقة جديدة تســـهل إجراء التكامل وبعد التكامل نعوض المتغير الأصـــلي في النتيجة كما في الأمثلة التالية

Example (16): Evaluate $\int \frac{dx}{x-\sqrt{x}}$

Let $u=\sqrt{x} \Rightarrow u^2 = x \Rightarrow 2udu = dx$

$$\int \frac{dx}{x-\sqrt{x}} = \int \frac{2udu}{u^2-u} = \int \frac{2udu}{u(u-1)} = 2\int \frac{du}{u-1} = 2\ln(u-1)+c = 2\ln(\sqrt{x}-1)+c$$

Example (17): Evaluate $\int \frac{dx}{1+\sqrt[4]{x}}$

Let $u = x^{1/4} \Rightarrow u^4 = x \Rightarrow 4u^3du = dx$

$$\int \frac{dx}{1+\sqrt[4]{x}} = \int \frac{4u^3du}{1+u} = 4\int \frac{(u^3+1-1)du}{1+u} = 4\int \frac{(u^3+1)du}{1+u} - 4\int \frac{du}{1+u}$$

$$= 4\int \frac{(u+1)(u^2-u+1)du}{1+u} - 4\int \frac{du}{1+u} = 4(\frac{u^3}{3}-\frac{u^2}{2}+u) - 4\ln(1+u)+c$$

$$= 4(\frac{x^{3/4}}{3}-\frac{x^{1/2}}{2}+x^{1/4}) - 4\ln(1+x^{1/4})+c$$

Example (18): Evaluate $\int_0^3 \sqrt{x+1}dx$

سنحل هذا المثال بطريقتين

- $\int_0^3 \sqrt{x+1}dx = \left.\frac{(x+1)^{3/2}}{3/2}\right]_0^3 = \frac{2}{3}(8-1) = \frac{14}{3}$

- $u = \sqrt{x+1} \Rightarrow u^2 = x+1 \Rightarrow u^2 - 1 = x \Rightarrow 2udu = dx$

  $x = 0 \Rightarrow u = \sqrt{x+1} = \sqrt{0+1} = 1$

  $x = 3 \Rightarrow u = \sqrt{x+1} = \sqrt{3+1} = 2$

  $\int_0^3 \sqrt{x+1}dx = \int_1^2 u(2u)du = 2\int_1^2 u^2du = 2\left[\frac{u^3}{3}\right]_1^2 = 2(\frac{8}{3}-\frac{1}{3}) = \frac{14}{3}$

Example (19): Evaluate $\int \frac{x+1}{x^2+2x+3}dx$

سنحل هذا المثال بطريقتين

- $\int \frac{x+1}{x^2+2x+3}dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+3}dx$

مشتقة المقام تساوي $2(x+1)$ وهي موجودة في البسط لذا نستخدم القاعدة $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

$$I = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+3}dx = \frac{1}{2}\ln|x^2+2x+3| + c$$

- Let $u = x^2+2x+3 \Rightarrow du = 2x+2$

$$I = \int \frac{x+1}{x^2+2x+3}dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x+3}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+2x+3| + c$$

Example (20): Evaluate $\int \tan^3 2x \sec^2 2x dx$

Let $u = \tan 2x \Rightarrow du = 2\sec^2 2x dx$

$$I = \int \tan^3 2x \sec^2 2x dx = \frac{1}{2}\int (\tan 2x)^3 (\sec^2 2x).(2)dx = \frac{1}{2}\int u^3 du = \frac{u^4}{8} + c = \frac{(\tan 2x)^4}{8} + c$$

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Exercise (2-1): Exe. (5.5) page 290 calculus-12

No.	Question	Answer
	Evaluating Indefinite Integrals Evaluate the indefinite integrals in Exercises 1-16 by using the given substitutions to reduce the integrals to standard form.	
1	$\int 2(2x+4)^5 dx,\ u = 2x+4$	$I = \frac{1}{6}(2x+4)^6 + c$
2	$\int 7\sqrt{7x-1}dx,\ u = 7x\text{-}1$	$I = \frac{2}{3}(7x-1)^{3/2} + c$
3	$\int 2x(x^2+5)^{-4}dx,\ u = x^2+5$	$I = \frac{-1}{3}(x^2+5)^{-3} + c$
4	$\int \frac{4x^3}{(x^4+1)^2},\ u = x^4+1$	$I = \frac{-1}{x^4+1} + c$
5	$\int (3x+2)(3x^2+4x)^4 dx,\ u = 3x^2+4x$	$I = \frac{1}{10}(3x^2+4x)^5 + c$
6	$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}dx,\ u = 1+\sqrt{x}$	$I = \frac{3}{2}(1+\sqrt{x})^{4/3} + c$

7	$\int \sin 3x dx$, $u=3x$	$I=\frac{-1}{3}\cos 3x+c$
8	$\int x\sin(2x^2)dx$, $u=2x^2$	$I=\frac{-1}{4}\cos(2x^2)+c$
9	$\int \sec 2t\cdot\tan 2t\cdot dt$, $u=2t$	$I=\frac{1}{2}\sec 2t+c$
10	$\int (1-\cos\frac{t}{2})^2\sin\frac{t}{2}dt$, $u=1-\cos\frac{t}{2}$	$I=\frac{2}{3}(1-\cos\frac{t}{2})^3+c$
11	$\int \frac{9r^2dr}{\sqrt{1-r^3}}$, $u=1-r^3$	$I=-6\sqrt{1-r^3}+c$
12	$\int 12(y^4+4y^2+1)^2(y^3+2y)dy$, $u=y^4+4y^2+1$	$I=(y^4+4y^2+1)^3+c$
13	$\int \sqrt{x}\sin^2(x^{3/2}-1)dx$, $u=x^{3/2}-1$	$I=\frac{1}{3}(x^{3/2}-1)-\frac{1}{6}\sin(2x^{3/2}-2)+c$
14	$\int \frac{1}{x^2}\cos^2(\frac{1}{x})dx$, $u=-\frac{1}{x}$	$I=\frac{-1}{2x}-\frac{1}{4}\sin(\frac{2}{x})+c$
15	$\int \csc^2 2\theta\cot 2\theta\cdot d\theta$ a. using $u=\cot 2\theta$ b. using $u=\csc 2\theta$	a. $I=\frac{-1}{4}\cot^2 2\theta+c$ b. $I=\frac{-1}{4}\csc^2 2\theta+c$
16	$\int \frac{dx}{\sqrt{5x+8}}$ a. using $u=5x+8$ b. using $u=\sqrt{5x+8}$	a. $I=\frac{2}{5}\sqrt{5x+8}+c$ b. $I=\frac{2}{5}\sqrt{5x+8}+c$
Evaluate the integrals in Exercises 17-50.		
17	$\int \sqrt{3-2s}ds$	$I=\frac{-1}{3}(3-2s)^{3/2}+c$
18	$\int \frac{1}{\sqrt{5s+4}}ds$	$I=\frac{2}{5}\sqrt{5s+4}+c$
19	$\int \theta\sqrt{1-\theta^2}d\theta$	$I=\frac{2}{5}(1-\theta^2)^{5/4}+c$
20	$\int 3y\sqrt{7-3y^2}dy$	$I=\frac{-1}{3}(7-3y^2)^{3/2}+c$
21	$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$	$I=\frac{-2}{1+\sqrt{x}}+c$
22	$\int \cos(3z+4)dz$	$I=\frac{1}{3}\sin(3z+4)+c$
23	$\int \sec^2(3x+2)dx$	$I=\frac{1}{3}\tan(3x+2)+c$
24	$\int \tan^2 x\cdot\sec^2 x\cdot dx$	$I=\frac{1}{3}\tan^3 x+c$
25	$\int \sin^5\frac{x}{3}\cdot\cos\frac{x}{3}\cdot dx$	$I=\frac{1}{2}\sin^6(\frac{x}{3})+c$
26	$\int \tan^7\frac{x}{2}\cdot\sec^2\frac{x}{2}\cdot dx$	$I=\frac{1}{4}\tan^8(\frac{x}{2})+c$

27	$\int r^2(\frac{r^3}{18}-1)^5 dr$	$I=(\frac{r^3}{18}-1)^6+c$
28	$\int r^4(7-\frac{r^5}{10})^3 dr$	$I=\frac{-1}{2}(7-\frac{r^5}{10})^4+c$
29	$\int \sqrt{x}\sin(x^{3/2}+1)dx$	$I=\frac{-2}{3}\cos(x^{3/2}+1)+c$
30	$\int \csc(\frac{v-\pi}{2})\cot(\frac{v-\pi}{2})\cdot dv$	$I=-2\csc(\frac{v-\pi}{2})+c$
31	$\int \frac{\sin(2t+1)}{\cos^2(2t+1)}dt$	$I=\frac{1}{2\cos(2t+1)}+c$
32	$\int \frac{\sec z\tan z}{\sqrt{\sec z}}dz$	$I=2\sqrt{\sec z}+c$
33	$\int \frac{1}{t^2}\cos(\frac{1}{t}-1)dt$	$I=-\sin(\frac{1}{t}-1)+c$
34	$\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)dt$	$I=2\sin(\sqrt{t}+3)+c$
35	$\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\sin\frac{1}{\theta}d\theta$	$I=-\frac{1}{2}\sin^2\frac{1}{\theta}+c$
36	$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}d\theta$	$I=-2\csc\sqrt{\theta}+c=-\frac{2}{\sin\sqrt{\theta}}+c$
37	$\int t^3(1+t^4)^3 dt$	$I=\frac{1}{16}(1+t^4)^4+c$
38	$\int \sqrt{\frac{x-1}{x^5}}dx$	$I=\frac{2}{3}(1-\frac{1}{x})^{3/2}+c$
39	$\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}dx$	$I=\frac{2}{3}(2-\frac{1}{x})^{3/2}+c$
40	$\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}dx$	$I=\frac{1}{3}(1-\frac{1}{x^2})^{3/2}+c$
41	$\int \sqrt{\frac{x^3-3}{x^{11}}}dx$	$I=\frac{2}{27}(1-\frac{3}{x^3})^{3/2}+c$
42	$\int \sqrt{\frac{x^4}{x^3-1}}dx$	$I=\frac{2}{3}(x^3-1)^{3/2}+c$
43	$\int x(x-1)^{10}dx$	$I=\frac{1}{12}(x-1)^{12}+\frac{1}{11}(x-1)^{11}+c$
44	$\int x\sqrt{4-x}dx$	$I=\frac{2}{5}(4-x)^{5/2}-\frac{8}{3}(4-x)^{3/2}+c$
45	$\int (x+1)^2(1-x)^5 dx$	$I=\frac{-1}{8}(1-x)^8+\frac{4}{7}(1-x)^7-\frac{2}{3}(1-x)^6+c$
46	$\int (x+5)(x-5)^{1/3}dx$	$I=\frac{3}{7}(x-5)^{7/3}+\frac{15}{2}(x-5)^{4/3}+c$
47	$\int x^3\sqrt{x^2+1}dx$	$I=\frac{1}{5}(x^2+1)^{5/2}-\frac{1}{3}(x^2+1)^{3/2}+c$
48	$\int 3x^5\sqrt{x^3+1}dx$	$I=\frac{2}{5}(x^3+1)^{5/2}-\frac{2}{3}(x^3+1)^{3/2}+c$

49	$\int \frac{x}{(x^2-4)^3}dx$	$I=-\frac{1}{4}(x^2-4)^{-2}+c$
50	$\int \frac{x}{(x-4)^3}dx$	$I=-(x-4)^{-1}-2(x-4)^{-2}+c$
If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 51 and 52.		
51	$\int \frac{18\tan^2 x\cdot\sec^2 x}{(2+\tan^3 x)^2}dx$ a. $u=\tan x,\ v=u^3,\ w=2+v$ b. $u=\tan^3 x,\ v=2+u$ c. $u=2+\tan^3 x$	a. $I=-\frac{6}{2+\tan^3 x}+c$ b. $I=-\frac{6}{2+\tan^3 x}+c$ c. $I=-\frac{6}{2+\tan^3 x}+c$
52	$\int \sqrt{1+\sin^2(x-1)}\sin(x-1)\cos(x-1)dx$ a. $u=x-1, v=\sin u,\ w=1+v^2$ b. $u=\sin(x-1),\ v=1+u^2$ c. $u=1+\sin^2(x-1)$	a. $I=\frac{1}{3}(1+\sin^2(x-1))^{3/2}+c$ b. $I=\frac{1}{3}(1+\sin^2(x-1))^{3/2}+c$ c. $I=\frac{1}{3}(1+\sin^2(x-1))^{3/2}+c$
Evaluate the integrals in Exercises 53 and 54.		
53	$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}dr$	$\frac{1}{6}\sin\sqrt{3(2r-1)^2+6}+c$
54	$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}d\theta$	$\frac{4}{\sqrt{\cos\sqrt{\theta}}}+c$

Exercise (2-2): By a simplifying substitution method evaluate the integrals.

Exe.(3-12) –page (398) - blue book   باستخدام طريقة ابدال المتغيرات، أوجد التكاملات.

No.	Question	Answer
1	$\int x^5\sqrt{2+x^3}dx$	$I=\frac{2}{15}(2+x^3)^{5/2}-\frac{4}{9}(2+x^3)^{3/2}+c$
2	$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx$	$I=-x+4\sqrt{x}-4\ln(1+\sqrt{x})+c$
3	$\int \frac{xdx}{3+\sqrt{1+2x}}$	$I=\frac{1}{6}(1+2x)^{3/2}-\frac{3}{4}(1+2x)+8\sqrt{1+2x}-12\ln(\sqrt{1+2x}+3)+c$
4	$\int \sqrt{\frac{1+x}{1-x}}dx$	$I=\sin^{-1}x-\sqrt{1-x^2}+c$
5	$\int \frac{xdx}{(2+x)^{1/4}}$	$I=\frac{4}{7}(2+x)^{7/4}-\frac{8}{3}(2+x)^{3/4}+c$
6	$\int \frac{dx}{x\sqrt{x+9}}$	$I=\frac{-2}{3}\tanh^{-1}\frac{\sqrt{x+9}}{3}+c$

7	$\int \frac{e^{2x}dx}{\sqrt{1+e^x}}$	$I=\frac{2}{3}(1+e^x)^{3/2}-2\sqrt{1+e^x}+c$
8	$\int \sqrt{x^3-x^2}\,dx$	$I=\frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+c$
9	$\int \frac{dx}{1+\sqrt{x}}$	$I=2\sqrt{x}-2\ln(1+\sqrt{x})+c$
10	$\int \frac{xdx}{(x^2+3)\sqrt{x^2-1}}$	$I=\frac{1}{2}\tan^{-1}(\sqrt{x^2-1})+c$
11	$\int x\sqrt{2x-1}dx$	$I=\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}+c$
12	$\int \frac{dx}{(x+4)\sqrt{x+5}}$	$I=-2\tanh^{-1}(\sqrt{x+5})+c$
13	$\int \frac{dt}{t-\sqrt{1-t^2}}$, $\int \frac{tdt}{1-\sqrt{1-t^2}}$	$I=\frac{1}{4}\ln[\cos(2\cos^{-1}t)]-\frac{1}{4}\ln[\sec(2\cos^{-1}t)+\tan(2\cos^{-1}t)]+\frac{1}{2}\cos^{-1}t+c$ $I=\sqrt{1-t^2}+\ln(1-\sqrt{1-t^2})+c$
14	$\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}dx$	$I=\frac{4}{5}x^{5/4}-x+\frac{4}{3}x^{3/4}-2x^{1/2}+4x^{1/4}-4\ln(x^{1/4}+1)+c$
15	$\int \frac{7x^3dx}{(1+x^4)^{1/2}}$	$I=\frac{7}{2}(1+x^4)^{1/2}+c$
16	$\int x^3\sqrt{x^2+a^2}dx$	$I=\frac{1}{5}(x^2+a^2)^{5/2}-\frac{a^2}{3}(x^2+a^2)^{3/2}+c$
17	$\int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}dx$	$I=2\left[2\ln(3)-\frac{1}{2}-2\ln(2)\right]=4\ln(3)-1-4\ln(2)$
18	$\int \frac{dx}{\sqrt{5x+8}}$	$I=\frac{2}{5}\sqrt{5x+8}+c$
19	$\int \frac{x}{\sqrt{x-1}}dx$	$I=\frac{-2}{3}\sqrt{1-x}(x+2)+c$
20	$\int \frac{\sqrt{x}}{x+1}dx$	$I=2\sqrt{x}-2\tan^{-1}(\sqrt{x})+c$
21	$\int \frac{xdx}{\sqrt{3x+1}}$	$I=\frac{2}{27}(3x-2)\sqrt{3x+1}+c$
22	$\int \frac{x+1}{\sqrt{2x-1}}dx$	$I=\frac{1}{3}(x+4)\sqrt{2x-1}+c$
23	$\int \frac{x-1}{\sqrt{x+4}}dx$	$I=\frac{2}{3}(x-11)\sqrt{x+4}+c$
24	$\int \frac{xdx}{\sqrt{4x+8}}$	$I=\frac{1}{3}(x-4)\sqrt{x+2}+c$

Exercise (2-3): By a simplifying substitution method or another method evaluate the integrals.

Exe.(32) - page 273-white book .باستخدام طريقة ابدال المتغيرات او طريقة اخرى، أوجد التكاملات

No.	Question	Answer
1	$\int (2x-1)^{3/2}dx$	$I=\frac{1}{5}(2x-1)^{5/2}+c$
2	$\int \sqrt{3x+2}dx$	$I=\frac{2}{3}(3x+2)^{3/2}+c$
3	$\int x\sqrt{16-x^2}dx$	$I=\frac{2}{3}(16-x^2)^{3/2}+c$
4	$\int \frac{dx}{(x-1)^3}$	$I=\frac{-1}{2(x-1)^2}+c$
5	$\int \frac{dx}{(3x+7)^3}$	$I=\frac{-1}{6(3x+7)^2}+c$
6	$\int \frac{2x+1}{x^2+x+1}dx$	$I=\ln(x^2+x+1)+c$
7	$\int \sin^2 x\cos xdx$	$I=\frac{(\sin x)^3}{3}+c$
8	$\int \tan 3x\sec 3xdx$	$I=\frac{1}{3}\sec 3x+c$
9	$\int x\sec^2(x^2)dx$	$I=\tan x^2+c$
10	$\int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)}$	$I=\ln(1+\sqrt{x})+c$
11	$\int e^x\cos(e^x)dx$	$I=\sin e^x+c$
12	$\int e^{2\cos 2x}\sin(2x)dx$	$I=\frac{e^{2\cos 2x}}{4}+c$
13	$\int \frac{x\ln(1+x^2)}{1+x^2}dx$	$I=\frac{\left(\ln(1+x^2)\right)^2}{4}+c$
14	$\int \frac{\sec^2 3x}{4+\cot 3x}dx$	$I=\frac{-1}{48}\ln(\cot 3x)-\frac{1}{12\cot 3x}+\frac{1}{48}\ln(\cot 3x+4)+c$
15	$\int \frac{dx}{(x-1)\sqrt{x+3}}$	$I=-\tanh^{-1}\sqrt{x+3}+c$
16	$\int_0^2 \frac{dx}{(x+1)^2}$	$I=\frac{2}{3}$
17	$\int_{-2}^{2}(x^3+1)^3x^2dx$	$I=\frac{4160}{12}$

18	$\int_0^2 x\sqrt{4-x^2}\,dx$	$I=\frac{8}{3}$

(19) جد المساحة بين المنحني $y=\frac{2x+1}{\sqrt{x^2+x+1}}$ ومحور السينات وبين المستقيمات $x=0$ ، $x=2$

الجواب : ($A=2(\sqrt{7}-1)$)

(20) جد المساحة بين المنحني $y=\tan x\sec^2 x$ ومحور السينات وبين المستقيمات $x=0$ ، $x=\pi/3$

الجواب : ($A=3/2$)