جامعة تكريت

كلية التربية للبنات

قسم الرياضيات

المرحلة : الثالثة

المادة : الاحصاء والاحتمالية

Moment of Random Variables

م. اسماء صالح قدوري

asmaa.salih@tu.edu.iq

Moment of Random Variables :

$Def :-Let\ a\ r.v.\ for\ c.r.v\ let\ K \in I^+\ Then\ E\,(X^k) is\ called$

$(the\ K^{th} moment\ of\ X)\ or\ (\ the\ moment\ of\ the\ ordedr\ \ K\ of\ X\ )$

$When\ K = 1 \ \rightarrow E\,(\,X^1) = 1^{st}\ moment\ of\ X = M \quad E\,(X^2)$
$= 2^{nd}\ moment\ of\ X$

$Note : E(X^K)\ exitis\ if\,f\ E\,(\,|X|^k) < \infty$

$Theorm\ "1":-\ If\ E\,(X^k\,)\ exsists\ then\ E\,(X^j)$

$existis\ ,j < K\ and\ j\,,k \in I^+$

$Proof\ : case\ "1"\ If\ (\,X\,) is\ c.r.v\ with\ p.d.f\ (\,X)\ since\ E\,(X^k)\ exsists$
$\rightarrow \therefore E\,(|X|^{\,k}\,) < \infty$

$T.P\ \ E(X^k) exsists\ ,T.P\ \ E\,(|X|^k) < \infty$

$$E\left(|X|^j\right) = \int_{-\infty}^{\infty} |X|^j\ f(x)dx\ .$$

$$E\left(\,|X|^j\right) = \int_{|X|\le 1}^{-\infty} |X|^j\, f(X)\ dx + \int_{|X|>1}^{\infty} |X|^j\ f(x)dx$$

$2^{nd} central\ \ moment\ of\ R.V.X\ is\ equal\ to\ V(X)$

$Ex :-let\ X\ be\ a\ r.v.s\,.t\,.E\,(X) = 1, \qquad E(X^2) = 2\ and\ \ E(X^3) = 5\,,$
$Find\ the\ 3^{rd}\ central\ \ moment\ \ of\ X$

$sol :-\ \ E\,[(X - M^3)] = E\{X^3 - 3MX^2 + 3M^2X - M^3\}$
$= E(X^3) - 3ME(X^2) + 3M^2E(X) - M^3$

$= 5 - 3.1.2 + 3.1.1 - 1 = 1$

$\boldsymbol{Moment\ Generation\ Function\ (M.g.f.).}$

Def : A moment of generation function($M.g.f$) of a random variable x is a function that determines all moments of X , denoted by $M_x(t)$, suppose that $t \in (-h,h)\ , h > 0$.

$If\ E\ [e^{tx}], exitis\ \forall\ t \in (-h,h)\ then\ M_x(t) = E\ [e^{tx}]\ , -h < t < h$

$There\ are\ two\ case\ of\ M_x\ (t)$

$Case\ \ (1)$

$If\ \ X\ is\ \ a\ \ discrete$ random variable $from\ \ a\ p\ .m\ .f\ \ , f\ (x)$

$$M_x(\,t) = E[e^{tx}] = \sum_{\forall x} e^{tx} f(x)$$

$Case\ (\,2)$

$If\ x\ is\ a\ continuous$ random variable $have\ \ a\ p.d.f\ , f\ (x)$

$$M_x(\,t) = E[t^{tx}] = \int_0^{\infty} e^{tx}\ f\ (x)dx$$

$Ex:\ \ Given\ \ a\ p.d.f\ ,\qquad f\ (\,x) = \begin{Bmatrix} e^{-x}\ for\ \ x > 0 \\ 0 \qquad\qquad o.w \end{Bmatrix}$

$Find\ M_x(t)$

$sol:\ \ M_x(\,t) = E\ (e^{tx}\,) = \int_0^{\infty} e^{tx}.e^{-x}\ dx = \int_0^{\infty} e^{-(1-t)x}\ dx$

This integration exist only when $(1-\text{t}) > o$, its mean $t < 1$.

$$M_x(\,t) = \frac{-1}{1-t} e^{-[1-t]x}|_0^{\infty} = \frac{-1}{1-t}\ [e^{-\infty} - e^0] = \frac{1}{1-t}$$

$$M_x(\,t) = \tfrac{1}{1-t} \qquad for\ t < 1.$$

$Theorem$ " 2": $Let\ x\ be\ a\ random\ variable\ have\ a\ M.g.f.,then$

$$M_x(t=0)=1\,,M_x'(t=0)=E(x)\,,M_x''(t=0)=E(x^2),$$

$$M_x'''(t=0)=E(x^3),\quad \dots\ M_x^h(t=0)=E(x^h)$$

proof:

$M_x(t)=E(e^{tx})$, $by\ Macclaurin\ series$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^k}{k!}+\cdots$$

$$\therefore e^{tx}=1+tx+\frac{t^2x^2}{2!}+\frac{t^3x^3}{3!}+\cdots+\frac{t^kx^k}{k!}+\cdots$$

$$M_x(t)=E[1+tx+\frac{t^2x^2}{2!}+\frac{t^3x^3}{3!}+\cdots+\frac{t^kx^k}{k!}+\cdots]$$

$$M_x(t)=1+tEx+\frac{t^2E(x^2)}{2!}+\frac{t^3E(x^3)}{3!}+\cdots+\frac{t^kE(x^k)}{k!}+\cdots]$$

$$M_x(t=0)=M_x(0)=1$$

$$M_x'(t)=\frac{dM_x(t)}{dt}$$

$$=E(x)+\frac{2t}{2!}E(x^2)+\frac{3t^2}{3!}E(x^3)+\frac{kt^{k-1}}{k!}E(x^k)+\cdots$$

$$M_x'(t=0)=E(x)$$

$$M_x''(t=0)=E(x^2)+\frac{6t}{3!}E(x^3)+\cdots+\frac{k(k-1)t^{k-2}}{k!}E(x^k)+\cdots$$

$$M_x''(t=0)=E(x^2)$$

similarly we can find $E(x^3),E(x^4),\dots,E(x^k)$

Note:

$$M_x(t) = M_x(0) + t\,M_x'(0) + \frac{t^2}{2!} \quad M_x''(0) + \cdots + \frac{t^k}{k!} M_x^k(t=0)$$

This series is called $M.g.f.\ by\ Macclaurin\ series.$

EX.: Given $M_x(t) = \frac{1}{1-2t}, t < \frac{1}{2}$, find $E(x) and\ V(x)$

sol.

$$M_x(t) = (1-2t)^{-1}, t < \frac{1}{2}$$

$$M_x'(t) = -(1-2t)^{-2}.(-2) = 2(1-2t)^{-2}$$

$$E(x) = M_x'(t=0) = 2(1-0)^{-2} = 2$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$M_x''(t) = 8(1-2t)^{-3}$$

$$E(x^2) = \quad M_x''(0) = 8(1-0)^{-3} = 8$$

$$\therefore V(x) = 8 - (2)^2 = 4 \geq 0$$

$Theorem$ "3":

$Let\ x\ be\ a\ random\ variable\ have\ a\ M.g.f. M_x(t),\ If\ y = ax + b,$

$$a, b \in R, Then\ M_x(t) = e^{bt} M_x(at)$$

proof:

$$y = ax + b$$

$$M_x(t) = E(\mathrm{e}^{\mathrm{t}y}) = E(\mathrm{e}^{t(ax+b)}]$$

$$= E\left[\mathrm{e}^{(\mathrm{at})\mathrm{x}}.\mathrm{e}^{\mathrm{bt}}\right] = \mathrm{e}^{\mathrm{bt}} E[\mathrm{e}^{(\mathrm{at})\mathrm{x}}]$$

$$= \mathrm{e}^{\mathrm{bt}} M_x(at)$$

since $M_x(t) = E(\mathrm{e}^{\mathrm{tx}})$

$$\rightarrow M_x(a\,t) = E[\mathrm{e}^{(\mathrm{at})\mathrm{x}}]$$