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Neyman Pearson Theorem

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Neyman Pearson Theorem:

Let $x_1, x_2, \dots \dots, x_n$, n is fixed positive integer denotes a random sample from a distribution that has a p.d.f. $f(x, \theta)$ then the joint p.d.f of $x_1, x_2, \dots \dots, x_n$ is :

$$L(\theta; x_1, x_2, \dots \dots, x_n) = f(x_1, \theta), f(x_2, \theta) \dots \dots f(x_n, \theta)$$

Let θ' and θ'' be destines fixed values of θ so that

$\omega = \{ \theta; \theta = \theta', \theta'' \}$ and let k be a positive number.

Let C be subset of the sample space such that :-

i- $\frac{L(\theta'; x_1, x_2, \dots\dots, x_n)}{L(\theta''; x_1, x_2, \dots\dots, x_n)} \le k$ for each point $x_1, x_2, \dots \dots, x_n \epsilon C$

ii- $\frac{L(\theta'; x_1, x_2, \dots\dots, x_n)}{L(\theta''; x_1, x_2, \dots\dots, x_n)} \ge k$ for each point $x_1, x_2, \dots \dots, x_n \epsilon \bar{C}$

iii- $\alpha = P[(x_1, x_2, \dots \dots, x_n) \in C / H_0]$

Then C is a best Critical region of size α for testing the simple hypothesis $H_0 = \theta = \theta'$ against $H_1 = \theta = \theta''$.

proof:

Let there are another Critical Region A with significance level α .

$$\int \dots \int L(\theta; x_1, x_2, \dots \dots, x_n) dx_1 dx_2 \dots dx_n = \int_R L(\theta)$$

We wish to show :-

$$\int_C L(\theta'') - \int_A L(\theta'') \ge 0$$

C =$((C \cap A) \cup (C \cap \bar{A})$

A= $(A \cap C) \cup (A \cap \bar{C})$

we found that

1- $\int_c L(\theta^{"}) - \int_A L(\theta^{"}) = \int_{c\cap A} L(\theta^{"}) + \int_{C\cap \bar{A}} L(\theta^{"}) -$
$\int_{A\cap C} L(\theta^{"}) - \int_{A\cap \bar{C}} L(\theta^{"})$

$= \int_{C\cap \bar{A}} L(\theta^{"}) - \int_{A\cap \bar{C}} L(\theta^{"})$

and from the theorem:-

$L(\theta') \leq kL(\theta^{"})$ for each point in C.

$L(\theta') > kL(\theta^{"})$ for each point in $\bar{C}$

Hint $L(\theta^{"}) \geq \frac{1}{k} L(\theta')$ at each point of $c \cap A$

$$\int_{C\cap \bar{A}} L(\theta^{"}) \geq \frac{1}{k} \int_{C\cap A} L(\theta')$$

But
$L(\theta^{"}) \leq \frac{1}{k} L(\theta')$ at each point of $\bar{C}$ and hence at each point of $A \cap \bar{C}$
accordingly .

$$\int_{C\cap \bar{A}} L(\theta^{"}) \leq \frac{1}{k} \int_{A\cap \bar{C}} L(\theta')$$

$$\int_{C\cap \bar{A}} L(\theta^{"}) - \int_{A-\bar{C}} L(\theta^{"}) \geq \frac{1}{K} \int_{C\cap \bar{A}} L(\theta') - \frac{1}{K} \int_{A\cap \bar{C}} L(\theta')$$

from 1

2- $\int_c L(\theta') - \int_A L(\theta^{"}) \geq \frac{1}{K} \int_{C\cap A} L(\theta') - \int_{A\cap \bar{C}} L(\theta')$

then

$$\int_{C\cap\bar{A}} L(\theta') - \int_{A\cap\bar{C}} L(\theta') = \int_{c\cap\bar{A}} L(\theta')$$

$$+ \int_{C\cap A} L(\theta') - \int_{A\cap C} L(\theta') - \int_{A\cap\bar{C}} L(\theta')$$

$=\alpha - \alpha = 0$

By substituting in 2

$$\rightarrow \int_{C} L(\theta^{"}) - \int_{A} L(\theta^{"}) \geq 0$$

$$\rightarrow (p.o.T\ in\ C) - (\ P.O.T\ in\ A) \geq 0$$

$$\therefore (P.O.T\ in\ C) \geq (\ P.O.T\ in\ A)$$

∴ C is best critical region.

Example :

Let X_i be a random variable which has a binomial distribution with n =5 and p= θ , let $H_0 = \theta = \frac{1}{2}$ and $H_1 = \theta = \frac{3}{4}$, find a best critical region of size.

i- $\alpha = \frac{1}{32}$ ii- $\alpha = \frac{6}{32}$

Sol : Since X ~ b (n , Q) = b (5 , Q)

∴ f (X , Q) = $\begin{cases} C_x^5\ Q^x\ (1 - Q^{5-x})\ , X = 0\ ,1,2\ , \dots .5 \\ 0 \qquad\qquad\qquad\qquad\qquad e.w. \end{cases}$

x	0	1	2	3	4	5
H_0:f(X,1/2)	1/32	5/32	10/32	10/32	5/32	1/32
H_1:f(X,3/4)	1/1024	15/1024	80/1024	27/1024	405/1024	243/1024

$$(i)\ \propto = \frac{1}{32}$$

$A_1 = \{x: x = 0\}$ لان قيمة الدالة

عند X=0 = 1/32 وهي نفس قيمة ∝

$$A_2 = \{x: x = 5\}$$

$$PH_0\,,(A_1) = \frac{1}{32} = \propto, PH_0(A_2) = \frac{1}{32} = \propto$$

$$PH_1(A_1) = PH_1(X = 0) = \frac{1}{1024}$$

$$PH_1(A_2) = PH_1(x = 5) = \frac{243}{1024}$$

$$\therefore PH_1(A_1) > PH_1(A_2)\ Thus\ A_2\ is\ B.C.R.of\ size \propto = \frac{1}{32}$$