جامعة تكريت

كلية التربية للبنات

قسم الرياضيات

المرحلة الثانية

مادة التفاضل المتقدم

Convergence and Divergence tests for Infinite Series

التقارب والتباعد للمتسلسلات غير المنتهية

اسم التدريسي : أ.م. ايلاف صباح عبدالواحـد

الايميل: elafs.math@tu.edu.iq

Ex: Find the sum of the following series

1) $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots, a = 5, r = \left|-\frac{1}{4}\right| < 1.$

$\therefore S = \frac{5}{1+\frac{1}{4}} = 4.$

2) $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$, since $\left|\frac{1}{2}\right| < 1, \left|\frac{1}{6}\right| < 1$, then $A = \frac{1}{1-\frac{1}{2}} = 2$,

$B = \frac{1}{1-\frac{1}{6}} = \frac{6}{5}$

Then $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right) = A - B = 2 - \frac{6}{5} = \frac{4}{5}$.

3) $\sum_{n=0}^{\infty} \frac{4}{2^n}, a = 4, r = \left|\frac{1}{2}\right| < 1, \therefore S = \frac{4}{1-\frac{1}{2}} = 8.$

4) $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - \cdots$, $\{S_n\} = \{1,0,1,0, \dots\}$ is Div. because of the oscillation of S_n between 1,0.

Theorem: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series then:

1) Sum Rule: $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B.$
2) Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA.$

Convergence and Divergence tests for Infinite Series

- **(1) The n[the]-term test for a divergent series.**

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is Div.

Notice: If $\lim_{n\to\infty} a_n = 0$, then we can't conclude that the series is convergent, this condition is necessary, but not sufficient for convergence.

Ex: Use the nth-term test to find whether the following series are divergent or not.

1) $\sum_{n=1}^{\infty} \frac{3n}{5n+1}$, since $\lim_{n\to\infty} \frac{3n}{5n+1} = \frac{3}{5} \neq 0$. Then is Div.

2) $\sum_{n=1}^{\infty} nSin(\frac{1}{n})$, Let $= \frac{1}{n}$, $\lim_{n\to\infty} \frac{Sin(u)}{u} = 1 \neq 0$. Div.

$$3)\sum_{n=1}^{\infty} \frac{n}{2n+5} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{2n}{n}+\frac{5}{n}} = \frac{1}{2+\frac{5}{n}} = \frac{1}{2+\frac{5}{\infty}} = \frac{1}{2+0} = \frac{1}{2}$$

is div.

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} (\frac{n}{n}+\frac{1}{n}) = \lim_{n\to\infty} (1+\frac{1}{n}) = 1+0 = 1$$

- **(2) The P-Series**

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called, the P-Series (P is real constant).

$*$ If $= 1 \Rightarrow a_n = \frac{1}{n}$, then by integral test $\int_1^{\infty} \frac{1}{x} dx = \infty$. Div.

$*$ If $p > 1 \Rightarrow a_n = \frac{1}{n^p}$ we have $\int_1^{\infty} \frac{1}{x^p} dx = \frac{x^{-p+1}}{-p+1} \Big|_1^{\infty} = \frac{1}{1-p}(0-1) = \frac{1}{p-1}$ .Conv.

$\therefore$ The P-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, and divergent if $p \leq 1$.

Ex: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \Rightarrow p = \frac{1}{2} < 1.$

The series is div. by p-series.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^3} \Rightarrow p = 3 > 1$ .The series is conv. by p-series.

- **(3) The Integral Test**

Corollary: A series $\sum a_n$ of non-negative terms converges if and only if it's partial sums are bounded from above.

Ex: The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, the harmonic series is divergent.

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{4}{8} = \frac{1}{2}\right) + \left(\frac{8}{16} = \frac{1}{2}\right) + \cdots$$

In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is greater than $\frac{2^n}{2^{n+1}} = \frac{1}{2}$. The sequence of partial sums is not bounded from above, if $n = 2^k$, the partial sum S_n is greater than $\frac{k}{2}$, the harmonic series is diverges.

Theorem: (The Integral test)

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ ($N \in Z^+$). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both convergent or both divergent.

Ex: Test the convergence and divergence of the series:

1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $f(x) = \frac{1}{x^2} \Rightarrow \int_1^{\infty} \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^{\infty} = 0 - 1 = 1$. The series is convergent.

2) $\sum_{n=1}^{\infty} \frac{lnn}{n}$, $f(x) = \frac{lnx}{x} \Rightarrow \int_1^{\infty} \frac{lnx}{x} dx = \frac{(lnx)^2}{2}\Big|_1^{\infty} = \infty$. Div.

3) $\sum_{n=1}^{\infty} \frac{1}{n^2+1} \Rightarrow \int_1^{\infty} \frac{1}{x^2+1} dx = \tan^{-1}(x)\Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. Conv.

Ex: Test the convergence and divergence of the series by integral test.

$$\sum_{n=1}^{\infty} \frac{e^{n}}{1+e^{2n}}$$

$$f(x) = \frac{e^{x}}{1+e^{2x}}$$

$$f'(x) = \frac{(1+e^{2x}).e^{x} - e^{x}.(2e^{2x})}{(1+e^{2x})^{2}} = \frac{e^{x}+e^{3x}-2e^{3x}}{(1+e^{2x})^{2}} = \frac{e^{x}-e^{3x}}{(1+e^{2x})^{2}} = \frac{e^{x}(1-e^{2x})}{(1+e^{2x})^{2}}$$

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$\forall \; x \geq 1$

$$\int_{1}^{\infty} f(x)\,dx = \lim_{x\to\infty} \int_{1}^{\infty} \frac{e^{x}}{1+e^{2x}}\,dx$$

$$\lim_{x\to\infty} \int_{1}^{\infty} \frac{e^{x}}{1+e^{2x}} \Rightarrow \lim_{x\Rightarrow\infty} \frac{e^{x}}{1+(e^{x})^{2}}\,dx$$

$$\lim_{x\to\infty}[\tan^{-1}(e^{x})]_{1}^{\infty} = \lim_{x\to\infty}[\tan^{-1}(e^{\infty}) - \tan^{-1}(e^{1})] = \frac{\pi}{2} - \tan^{-1}(e^{1})$$

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