Closed Methods for Solving Nonlinear Equations

Lecture Notes

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Chapter 1

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1.1 Introduction

Nonlinear algebraic equations are wide spread in science and engineering and therefore their solutions are important in scientific applications. There are a glut of numerical methods for solving these equations, and in these lecture notes, we study the most commonly used ones such as bisection, secant and Newton methods. Locating positions of roots of nonlinear equations is a topic of great importance in numerical analysis. The problem under consideration maybe has a root or has no root at all. The numerical methods which are used to find the roots of nonlinear equations are called **root-finding algorithms** or **numerical methods for locating a root**. **Iteration** is an important and basic concept in both *mathematics* and *computer science*, and has applications in *physics* and *engineering*. In these notes we consider a class of methods called **iterative methods** or **iteration methods** or **recursive methods** and as the name indicates a process is repeated until an acceptable solution is obtained.

Definition 1 (Zero of a Function)**.** *Let f be a real or complex valued function of a real or complex variable x. A real or complex number r satisfies $f(r) = 0$ is called a* ***zero of*** *f or also called a* ***root of equation*** *$f(r) = 0$.*

Definition 2 (Order of a Zero)**.** *Let f and its derivatives $f', f'', \cdots, f^{(M)}$ are continuous and defined on an interval about the zero $x = r$. The function f or the equation $f(x) = 0$ is said to be has a zero or a root of order $M \geq 1$ at $x = r$ if and only if*

$$f(r) = 0,\ f'(r) = 0,\ f''(r) = 0, \cdots, f^{(M-1)}(r) = 0,\ f^{(M)} \neq 0. \tag{1.1}$$

*If $M = 1$ then r is called a **simple zero** or a **simple root**, and if $M > 1$ it is called a **multiple zero** or a **multiple root**. A zero (root) of order $M = 2$ is called a **double zero (root)**, and so on. Also, the zero (root) of order M is called a **zero (root) of multiplicity** M.*

Lemma 1. *If the function f has a zero r of multiplicity M, then there exists a continuous function h such that f can be factorised as*

$$f(x) = (x - r)^M h(x), \quad \lim_{x \to r} h(x) \neq 0. \tag{1.2}$$

Theorem 1 (Simple Zero Theorem)**.** *Assume that $f \in C^1[a, b]$. Then, f has a simple zero at $r \in (a, b)$ if and only if $f(r) = 0$ and $f'(r) \neq 0$.*

Example 1. *The function $f(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$ has two real zeros $r_1 = 2$ and $r_2 = 3$, whereas the corresponding equation $x^2 - 5x + 6 = (x - 2)(x - 3) = 0$ has two real roots $r_1 = 2$ and $r_2 = 3$. According to the Lemma 1, the function f is factorised to*

$$f(x) = (x - 2)^1(x - 3) \text{ or } f(x) = (x - 3)^1(x - 2)$$

Example 2. *Show that the function $f(x) = e^{2x} - x^2 - 2x - 1$ has a zero of multiplicity 2 (double zero) at $x = 0$.*

Solution:

$$f(x) = e^{2x} - x^2 - 2x - 1, \; f'(x) = 2e^{2x} - 2x - 2, \text{ and } f''(x) = 4e^{2x} - 2.$$

Hence,

$$f(0) = e^0 - 0 - 0 - 1 = 0, \; f'(0) = 2e^0 - 0 - 2 = 0, \text{ and } f''(0) = 4e^0 - 2 = 2 \neq 0,$$

so, this implies that f has a double zero at $x = 0$.

1.2 Closed Methods

The basic idea of these methods is to find a closed interval $[a, b]$ no mattar how large such that it contains the root of the equation $f(x) = 0$ by stipulating that $f(a)$ and $f(b)$ have opposite signs, and for this reasons they are called **closed methods**. Also, these methods are known as **bracketing methods**. Once the interval is determined an iterative process is started until we reach a sufficiently small interval around the root for this reason these methods are termed **globally convergent methods**.

1.2.1 Bisection Method

It is a bracketing method used to find a zero of a continuous function f on the initial interval $[a, b]$ where a and b are real numbers, i.e. to find x such that $f(x) = 0$, this method requires that $f(a)$ and $f(b)$ have different signs. This method is based on the Intermediate Value Theorem, since f is continuous and has opposite signs on $[a, b]$ then there is a number r in $[a, b]$ such that $f(r) = 0$. This method is also known as **Bolzano method** or **bisection method of Bolzano** or **binary search method** or **interval halving method**. The first step in the solution process is to compute the midpoint $c = (a + b)/2$ of the interval $[a, b]$ and to proceed we consider the three cases:

1. If $f(a)f(c) < 0$ then r lies in $[a, c]$.
2. If $f(c)f(b) < 0$ then r lies in $[c, b]$.
3. If $f(c) = 0$ then the root is $c = r$.

To begin, set $a_1 = a$ and $b_1 = b$ and let $c_1 = \frac{a_1+b_1}{2}$ be the midpoint of the interval $[a_1, b_1] = [a, b]$.

- If $f(c_1) = 0$ then the root is $r = c_1$.
- If $f(c_1) \neq 0$ then either $f(a_1)f(c_1) < 0$ or $f(c_1)f(b_1) < 0$.
  - (i). If $f(a_1)f(c_1) < 0$ then r lies in $[a_1, c_1]$, and we squeeze the interval form the right and set $a_2 = a_1$ and $b_2 = c_1$, i.e. $[a_2, b_2] = [a_1, c_1]$.
  - (ii). If $f(c_1)f(b_1) < 0$ then r lies in $[c_1, b_1]$, and and we squeeze the interval form the left and set $a_2 = c_1$ and $b_2 = b_1$, i.e. $[a_2, b_2] = [c_1, b_1]$.
  - (iii). Compute $c_2 = \frac{a_2+b_2}{2}$ the midpoint of the interval $[a_2, b_2]$.
  - (iv). Then, we proceed in this way until we reach the nth interval $[a_n, b_n]$ and then compute its midpoint $c_n = \frac{a_n+b_n}{2}$.
- Finally, construct the interval $[a_{n+1}, b_{n+1}]$ which brackets the root and its midpoint $c_{n+1} = \frac{a_{n+1}+b_{n+1}}{2}$ will be an approximation to the root r.

In the bisection method the initial interval $[a, b]$ is bisected and the interval width is decreased by half each time until we reach an arbitrarily small interval that brackets the root and we take the midpoint of this final interval as a reasonable approximation of the root r.

Remark 1. *(a). The interval* $[a_{n+1}, b_{n+1}]$ *is wide as half as the interval* $[a_n, b_n]$ *i.e. the width of each interval is as half as the width of the previous interval. Let* $\{\ell_n\}$ *be a sequence of widths of intervals* $[a_n, b_n]$*, i.e.* $\ell_n = \frac{b_n - a_n}{2}$*,* $n = 1, 2, \cdots$*. Hence* $\lim_{n\to\infty} \ell_n = 0$*, where* ϵ *is the preassigned value of the error (**tolerance**) i.e.*

$$|r_{n+1} - r_n| \leq \epsilon, \ n = 0, 1, \cdots . \tag{1.3}$$

(b). The sequence of left endpoints a_n*,* $n = 1, 2, \cdots$*, is increasing and the sequence* b_n*,* $n = 1, 2, \cdots$ *of right endpoints is decreasing i.e.*

$$a_0 \leq a_1 \leq \cdots \leq a_n \leq \cdots \leq r \leq \cdots \leq b_n \leq \cdots \leq b_1 \leq b_0. \tag{1.4}$$

Theorem 2 (**Bisection Method Theorem**). *Let* $f \in C[a, b]$ *such that* $f(a)f(b) < 0$ *and that there exists a number* $r \in [a, b]$ *such that* $f(r) = 0$*, and* $\{c_n\}$ *a sequence of the midpoints of intervals* $[a_n, b_n]$ *constructed by the bisection method, then the error in approximating the root* r *in the nth step is:*

$$|e_n| = |r - c_n| \leq \frac{b - a}{2^{n+1}}, \quad n = 0, 1, \cdots . \tag{1.5}$$

Hence, the sequence $\{c_n\}$ *is convergent and its limit is the root* r*, i.e.*

$$\lim_{n\to\infty} c_n = r. \tag{1.6}$$

The error bound in (1.5) *can be used to evaluate the required predetermined accuracy of the method.*

Proof. For Proof see the References or have a look in any standard numerical analysis text. □

Remark 2.
- *The number* N *of repeated bisections required to compute the nth approximation (midpoint)* c_n *of the root* r *is:*

$$N = int\Big(\frac{ln(b - a) - ln(\epsilon)}{ln(2)}\Big). \tag{1.7}$$

The formula (1.7) *is obtained from the error bound formula* (1.5).

- *The width of the nth interval* $[a_n, b_n]$ *is:*

$$|b_n - a_n| = \frac{|b_0 - a_0|}{2^n}. \tag{1.8}$$

Example 3. *(a) Use bisection method to show that* $f(x) = x\sin(x) - 1 = 0$ *has a real root in* $[0.5, 1.5]$. *Compute eleven approximations (i.e. use* $n = 10$*) to the root.*

(b) Evaluate the number of computations N *required to ensure that the error is less than the preassigned value (error bound)* $\epsilon = 0.001$.

Solution:
(a) We start with initial interval $[a0, b0] = [0.5, 1.5]$ and compute $f(0.5) = -0.76028723$ and $f(1.5) = 0.49624248$. We notice that $f(a_0)$ and $f(b_0)$ have opposite signs and hence, there is a root in the interval $[0.5, 1.5]$. Compute the midpoint $c_0 = \frac{a_0+b_0}{2} = \frac{0.5+1.5}{2} = 1$ and $f(1) = -0.15852902$. The function changes sign on $[c_0, b_0] = [1, 1.5]$, so, we set $[a_1, b_1] = [c_0, b_0] = [1, 1.5]$, and compute the midpoint $c_1 = \frac{a_1+b_1}{2} = \frac{1+1.5}{2} = 1.25$ and $f(1.25) = 0.18623077$. Hence, the root lies in the interval $[a_1, c_1] = [1, 1.25]$. Set $[a_2, b_2] = [a_1, c_1] = [1, 1.25]$ and continue until we compute $c_{10} = 1.11376953125$. The details are explained in Table 1.1.

n	Left Endpoint a_n	Midpoint c_n	Right Endpoint b_n	Function Value $f(c_n)$
0	0.5	1	1.5	-0.15852902
1	1	1.25	1.5	0.18623077
2	1	1.125	1.25	0.01505104
3	1	1.0625	1.125	-0.07182663
4	1.0625	1.09375	1.125	-0.02836172
5	1.09375	1.109375	1.125	-0.00664277
6	1.109375	1.1171875	1.125	0.00420803
7	1.109375	1.11328125	1.1171875	-0.00121649
8	1.11328125	1.115234375	1.1171875	0.00149600
9	1.11328125	1.1142578125	1.115234375	0.00013981
10	1.11328125	1.11376953125	1.1142578125	-0.00053832

Table 1.1: Bisection Method Solution of Example 3

(b)

$$N = int\Big(\frac{ln(1.5 - 0.5) - ln(0.001)}{ln(2)}\Big) = int\Big(\frac{ln(1) - ln(0.001)}{ln(2)}\Big) =$$

$$int\Big(\frac{0 - (-6.90775528)}{0.69314718}\Big) = int\Big(\frac{6.90775528}{0.69314718}\Big) = int(9.96578429) = 10.$$

1.2.2 False-Position Method

It also known as **regula falsi method**, it is similar to the bisection method in requiring that $f(a)$ and $f(b)$ have opposite signs. This method uses the abscissa of the point $(c, 0)$ at which the secant line called it SL joining the points $(a, f(a))$ and $(b, f(b))$ crosses the x-axis instead of using the midpoint of the interval as approximation of the zero of the function f as in the bisection method. To evaluate c, we need to compute the slope of line SL between the two points $(a, f(a))$ and $(b, f(b))$:

$$m = \frac{f(b) - f(a)}{b - a}.$$

Now, compute the slope of line SL between the two points $(c, f(c)) = (c, 0)$ and $(b, f(b))$:

$$m = \frac{f(b) - f(c)}{b - c} = \frac{f(b) - 0}{b - c} = \frac{f(b)}{b - c}.$$

By equating the two slopes, we obtain

$$\frac{f(b) - f(a)}{b - a} = \frac{f(b)}{b - c} \quad \Longrightarrow \quad c = b - \frac{f(b)(b - a)}{f(b) - f(a)}.$$

Now, we have the same possibilities as in the bisection method:

- If $f(c_0) = 0$ then the root is $r = c_0$.
- If $f(c_0) \neq 0$ then either $f(a_0)f(c_0) < 0$ or $f(c_0)f(b_0) < 0$.
  - (i). If $f(a_0)f(c_0) < 0$ then r lies in $[a_0, c_0]$, and set $a_1 = a_0$ and $b_1 = c_0$, i.e. $[a_1, b_1] = [a_0, c_0]$.
  - (ii). If $f(c_0)f(b_0) < 0$ then r lies in $[c_0, b_0]$, and set $a_1 = c_0$ and $b_1 = b_0$, i.e. $[a_1, b_1] = [c_0, b_0]$.
  - (iii). Compute $c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)}$.
  - (iv). Then, we proceed in this way until we reach the nth interval $[a_n, b_n]$ and then compute $c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$.

The general formula of the false-position method is

$$c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}, \ n = 1, 2, \cdots, \tag{1.9}$$

start with an initial interval $[a_1, b_1] = [a, b]$ such that f has opposite signs on it, then the sequence $\{c_n\}_{n=1}^{\infty}$ of successive approximations of the root converges to the root r of the equation $f(x) = 0$. In general, the false-position method is faster than the bisection method. Note that the interval width $b_n - a_n$ is getting smaller as n gets larger but it is not necessarily to approaches zero. For example, if the curve of the function $y = f(x)$ is concave near the point $(r, 0)$ where the graph of the function crosses the x-axis, then one of the endpoints of the interval is fixed and the other endpoint moves to the root. The fixed endpoint is called the **stagnant endpoint**.

Example 4. *Show that* $f(x) = 2x^3 - x^2 + x - 1 = 0$ *has at least on root in* $[0, 1]$.

Solution:
Since $f(0) = -1$ and $f(1) = 1$, then Intermediate Value Theorem implies that this continuous function has a root in $[0, 1]$. Set $[a_0, b_0] = [0, 1]$ and compute $c_0 = b_0 - \frac{f(b_0)(b_0-a_0)}{f(b_0)-f(a_0)} = f(1) - \frac{f(1)(1-0)}{f(1)-f(0)} = 1 - \frac{1(1-0)}{1-(-1)} = 0.5$, also compute $f(c_0) = f(0.5) = -0.5$. Hence, the root lies in $[c_0, b_0]$ we squeeze from the left and set $a_1 = c_0 = 0.5$ and $b_1 = b_0 = 1$, to have $[a_1, b_1] = [0.5, 1]$. Now, compute the new approximation to the root $c_1 = b_1 - \frac{f(b_1)(b_1-a_1)}{f(b_1)-f(a_1)} = f(1) - \frac{f(1)(1-0.5)}{f(1)-f(0.5)} = 1 - \frac{1(1-0.5)}{1-(-0.5)} = 2/3 \approx 0.66666667$, $f(c_1) = -0.18518519$. The function has opposite signs on the interval $[c_1, b_1]$, set $a_2 = c_1 = 0.66666667$ and $b_2 = b_1 = 1$. so we have $[a_2, b_2] = [0.66666667, 1]$. Continue this we and stop at $c_7 = 0.73895443$.

n	a_n	c_n	b_n	$f(c_n)$
0	0	0.5	1	-0.5
1	0.5	0.66666667	1	-0.18518519
2	0.66666667	0.71875000	1	-0.05523681
3	0.71875000	0.73347215	1	-0.01532051
4	0.73347215	0.73749388	1	-0.00416160
5	0.73749388	0.73858180	1	-0.00112399
6	0.73858180	0.73887530	1	-0.00030311
7	0.73887530	0.73895443	1	-0.00008171

Table 1.2: False Position Method Solution of Example 4

Exercises

Exercise 3. *Solve Example 4 using the bisection method and compare the solution with false position method's solution of the same problem.*

Exercise 4. *Repeat solving Example 3 using the false position method and compare the results with the solution of the bisection method for the same problem.*

Exercise 5. *Let* $f(x) = x^2 - 5$ *and* $r_0 = 1.5$. *Use bisection and false position methods to find* r_7 *the approximation to the positive root* $r = \sqrt{5}$.