Direct Methods for Solving Systems of Linear Equations

Lecture Notes

Mohammad Sabawi

Department of Mathematics
College of Education for Women
Tikrit University

Email: mohammad.sabawi@tu.edu.iq

01 October 2021

Contents

Chapter 1

Direct Methods for Solving Systems of Linear Equations

1.1 Introduction

Many phenomena and relationships in nature and real life applications are linear, meaning that results and their causes are proportional to each other. Solving linear algebraic equations is a topic of great importance in numerical analysis and other scientific disciplines such as engineering and physics. Solutions to Many problems reduced to solve a system of linear equations. For example, in finite element analysis a solution of a partial differential equation is reduced to solve a system of linear equations.

1.2 Direct Methods

Direct methods are techniques used for solving and obtaining the exact solutions (in theory) of linear algebraic equations in a finite number of steps. The main widely used direct methods are **Gaussian elimination method** and **Gauss-Jordan method**.

Consider the following linear system of dimension $n \times (n+1)$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

This system can be written in concise form by using matrix notation as $AX = B$ as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where $A_{n\times n}$ is square matrix and is called a **coefficient matrix**, $B_{n\times 1}$ is a column vector known as the **right hand side vector** and $X_{n\times 1}$ is a column vector known as **unknowns vector**. Also, this system can be written as

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{12} & \cdots & a_{1n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{12} & \cdots & a_{1n} & b_n \end{array}\right],$$

where $[A|B]$ is called the **augmented matrix**.

1.2.1 Backward Substitution Method

Backward substitution also called **back substitution** is an algorithm or technique used for solving **upper-triangular systems** which are systems such that their coefficient matrices are upper-triangular matrices. Assume that we have the following upper-triangular system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n-1}x_{n-1} + a_{1n}x_n &= b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n-1}x_{n-1} + a_{2n}x_n &= b_2 \\ a_{33}x_3 + \cdots + a_{3n-1}x_{n-1} + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{n-1n-1}x_{n-1} + a_{n-1n}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n. \end{aligned}$$

To find a solution to this system we follow the following steps provided that $x_{rr} \neq 0,\ r = 1, 2, \cdots, n$:

(1) Solve the last (nth) equation for x_n:

$$x_n = \frac{b_n}{a_{nn}}.$$

(2) Substitute x_n in the next-to-last $((n-1)th)$ equation and solve it for x_{n-1}:

$$x_{n-1} = \frac{b_{n-1} - a_{n-1n}x_n}{a_{n-1n-1}}.$$

(3) Now, x_n and x_{n-1} are known and can be used to find x_{n-2}:

$$x_{n-2} = \frac{b_{n-2} - a_{n-1n-1}x_{n-1} - a_{n-1n}x_n}{a_{n-2n-2}}.$$

(4) Continuing in this way until we arrive at the general step:

$$x_r = \frac{b_r - \sum_{j=r+1}^{n} a_{rj}x_j}{a_{rr}}, \quad r = n-1, n-2, \cdots 1.$$

Example 1. *Solve the following linear system using back substitution method*

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= 10 \\ x_2 - x_3 + 2x_4 &= 9 \\ 3x_3 - x_4 &= 1 \\ 3x_4 &= 6 \end{aligned}$$

Solution: Solve the last equation for x_4 to obtain

$$x_4 = \frac{6}{3} = 2.$$

Substitute $x_4 = 2$ in the third equation, we have

$$x_3 = \frac{1 + x_4}{3} = \frac{1+2}{3} = \frac{3}{3} = 1.$$

Now, use values $x_3 = 1$ and $x_4 = 2$ in the second equation to find x_2

$$x_2 = 9 + x_3 - 2x_4 = 9 + 1 - 4 = 6.$$

Finally, solve the first equation for x_1 yields

$$x_1 = \frac{10 - 2x_2 + x_3 - x_4}{3} = \frac{10 - 12 + 1 - 2}{3} = \frac{-3}{3} = -1.$$

Example 2. *Show that the following linear system has no solution*

$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= 10 \\ 0x_2 - x_3 + 2x_4 &= 9 \\ 3x_3 - x_4 &= 1 \\ 3x_4 &= 6 \end{aligned}$$

Solution: Solve the last equation for x_4 to obtain

$$x_4 = \frac{6}{3} = 2.$$

Substitute $x_4 = 2$ in the third equation, we have

$$x_3 = \frac{1 + x_4}{3} = \frac{1+2}{3} = \frac{3}{3} = 1.$$

Also, from the second equation we have

$$x_3 = 9 - 2x_4 = 9 - 4 = 6.$$

This contradiction implies that the linear system in above has no solution.

Example 3. *Show that the following linear system has infinitely many solutions*

$$\begin{aligned} 3x_1 + 3x_2 - x_3 + x_4 &= 10 \\ 0x_2 + x_3 + 0x_4 &= 1 \\ 3x_3 - x_4 &= 1 \\ 3x_4 &= 6 \end{aligned}$$

Solution: Solve the last equation for x_4 to obtain

$$x_4 = \frac{6}{3} = 2.$$

Substitute $x_4 = 2$ in the third equation, we have

$$x_3 = \frac{1 + x_4}{3} = \frac{1+2}{3} = \frac{3}{3} = 1.$$

Also, from the second equation we have

$$x_3 = 1.$$

Solve the first equation for x_2 yields

$$x_2 = \frac{10 - 3x_1 + x_3 - x_4}{3} = \frac{10 - 3x_1 + 1 - 2}{3} = \frac{9 - 3x_1}{3} = 3 - x_1.$$

Note that the equation for x_2 has infinitely many solutions since it depends upon x_1 which takes infinitely many values. Now, let $x_1 = 1$, we have $x_2 = 2$. Hence the solution set of the system is:

$$x_1 = 1,\ x_2 = 2,\ x_3 = 1,\ x_4 = 2.$$

1.2.2 Forward Substitution Method

Forward substitution is an algorithm or technique used for solving **lower-triangular systems** which are systems such that their coefficient matrices are lower-triangular matrices.

$$\begin{aligned}
&a_{11}x_1 &&= b_1\\
&a_{21}x_1 + a_{22}x_2 &&= b_2\\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &&= b_3\\
& &&\vdots\\
&a_{n-11}x_1 + a_{n-12}x_2 + a_{n-13}x_3 + \cdots + a_{n-1n-1}x_{n-1} &&= b_{n-1}\\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn-1}x_{n-1} + a_{nn}x_n &&= b_n.
\end{aligned}$$

To find a solution to this system we follow the following steps provided that $x_{rr} \neq 0,\ r = 1, 2, \cdots, n$:

(1) Solve the first $(1st)$ equation for x_1:

$$x_1 = \frac{b_1}{a_{11}}.$$

(2) Substitute x_1 in the second equation $(2nd)$ equation and solve it for x_2:

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}.$$

(3) Now, x_1 and x_2 are known and can be used to find x_3:

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}.$$

(4) Continuing in this way until we arrive at the general step:

$$x_r = \frac{b_r - \sum_{j=1}^{r-1} a_{rj} x_j}{a_{rr}}, \quad r = 2, 3, \cdots n.$$

Example 4. *Use the forward substitution method for solving the following linear system*

$$\begin{aligned} &4x_1 &&= 8 \\ &2x_1 + x_2 &&= -1 \\ &x_1 - x_2 + 5x_3 &&= 0.5 \\ &0.1x_1 + 2x_2 - x_3 + 2x_4 &&= 2 \end{aligned}$$

,

Solution: Solving the first equation for x_1 yields

$$x_1 = \frac{8}{4} = 2.$$

Using the value of x_1 to find x_2

$$x_2 = \frac{-1 - 2x_1}{2} = \frac{-1 - 2(2)}{2} = -2.5.$$

Use x_1 and x_2 to find x_3

$$x_3 = \frac{0.5 - x_1 + x_2}{5} = \frac{0.5 - 2 - 2.5}{5} = \frac{-4}{5} = -0.8.$$

Finally, solve for x_4 to have

$$x_4 = \frac{2 - 0.1x_1 - 2x_2 + x_3}{2} = \frac{2 - 0.1(2) - 2(-2.5) - 0.8}{2} = \frac{6}{2} = 3.$$

Example 5. *Show that there is no solution to the linear system*

$$\begin{aligned} &4x_1 &&= 8 \\ &2x_1 + x_2 &&= -1 \\ &x_1 - x_2 + 0x_3 &&= 0.5 \\ &0.1x_1 + 2x_2 - x_3 + 2x_4 &&= 2 \end{aligned}$$

,

Solution: Solving the first equation for x_1 yields

$$x_1 = \frac{8}{4} = 2.$$

Using the value of x_1 in the second equation to find x_2

$$x_2 = \frac{-1 - 2x_1}{2} = \frac{-1 - 2(2)}{2} = -2.5.$$

From the third equation we have

$$x_2 = x_1 - 0.5 = 2 - 0. = 1.5.$$

This contradiction indicates that there is no solution to the system in above.

Example 6. *Show that there are infinitely many solution to the following linear system*

$$\begin{aligned} &4x_1 &&= 8 \\ &2x_1 + x_2 &&= -1 \\ &0x_1 - x_2 + 0x_3 &&= 2.5 \\ &0.1x_1 + 2x_2 - x_3 + 2x_4 &&= 2 \end{aligned}$$

,

Solution: Solving the first equation for x_1 yields

$$x_1 = \frac{8}{4} = 2.$$

Using the value of x_1 in the second equation to find x_2

$$x_2 = \frac{-1 - 2x_1}{2} = \frac{-1 - 2(2)}{2} = -2.5.$$

From the third equation we have

$$x_2 = -2.5.$$

Solving the last equation for x_3 we obtain

$$x_3 = -2 + 0.1x_1 + 2x_2 + 2x_4 = -2 + 0.1(2) + 2(-2.5) + 2x_4 = 2x_4 - 6.8,$$

which has infinitely many solutions. Hence, the above linear system has infinitely many solutions. If we choose $x_4 = 8$, then we get $x_3 = -0.8$. So, the solution set is:

$$x_1 = 2,\ x_2 = -2.5,\ x_3 = 9.8,\ x_4 = 8.$$

1.2.3 Gaussian Elimination Method

Gaussian elimination method is also known as **Gauss elimination method** or simply **elimination method**. It is a direct method used for solving a system of linear algebraic equations. In this method we transform the linear system to an equivalent upper or lower triangular system and then solve it by backward or forward substitution. The process of transforming the linear system to an equivalent upper or lower triangular system is called **trianguarisation**.

Definition 1 (Equivalent Systems)**.** *Two linear algebraic systems of dimension $n \times n$ is said to be they are **equivalent** if they have the same solution sets.*

Definition 2 (Elementary Transformations)**.** *The following operations performed on a linear system transform it to an equivalent system:*

- ***Interchanges:*** *Changing the order of any two equations in the system.*
- ***Replacement:*** *Any equation of the system can be replaced by itself and a nonzero multiple of any other equation in the system.*
- ***Scaling:*** *Multiplying any equation in the system by a nonzero real constant.*

Definition 3 (Elementary Row Operations)**.** *The following operations performed on a linear system transform it to an equivalent system:*

- ***Interchanges:*** *Changing the order of any two rows in the matrix.*
- ***Replacement:*** *Any row in the matrix can be replaced by its sum and a nonzero multiple of any other row in the matrix.*
- ***Scaling:*** *Multiplying any row in the matrix by a nonzero real constant.*

Pivoting

Pivoting is an important process used ins solving linear systems in conjunction with Gaussian elimination and there different types of pivoting strategies as outlined below:

1. **Trivial Pivoting:** The process of using the element (entry) a_{kk} in the coefficient matrix A to eliminate the entries $a_{rk},\ r = k+1, k+2, \cdots n$ is called **pivoting process**. The element a_{kk} is called **pivotal element** and the kth row is called **pivotal row**. If the entry $a_{kk} = 0$,

then the row k cannot be used to eliminate the entries a_{rk}, $r = k + 1, k + 2, \cdots, n$ and we need to find a row r such that $a_{rk} \neq 0$, $r > k$, and then interchange the row k and the row r such that the pivotal element is nonzero. This process is called the **trivial pivoting**, also, if no interchange or switching between the rows is performed then the process is called only **pivoting** or **trivial pivoting**.

2. **Partial Pivoting:** To reduce the round-off errors or propagation of errors it is advisable to search for the the greatest element in the magnitude in column r that lies on or below the main diagonal, and then move it to the main diagonal in the pivotal row r to be the pivotal element and use it to eliminate the entries in the column r below the main diagonal, this process is called the **partial pivoting**. Determine row k below the main diagonal in which there is the largest element in the absolute value as follows:

$$a_{kr} = \max\{|a_{rr}|, |a_{r+1r}|, \cdots, |a_{n-1r}|, |a_{nr}|\}, \tag{1.1}$$

and then interchange the row k and row r for $k > r$. Now, since the entry in the main diagonal has the larges absolute value then the values of all the multipliers are:

$$|m_{kr}| \leq 1, k = r + 1, r + 2, \cdots, n,$$

and this will be helpful to keep the magnitudes of elements in the current matrix are relatively the same magnitudes of the elements in the original coefficient matrix.

3. **Scaled Pivoting:** In this approach, the pivoting element is chosen to be the largest in magnitude relative to the elements which lie in the same row. This type of pivoting is used when the entries in the same row vary largely in magnitude.

4. **Complete Pivoting:** In this technique, we use both partial and scaled pivoting and is sometimes referred to as **scaled partial pivoting** or **equilibrating**. In this process, we search all the entries in the column r that lie on or below the main diagonal for the largest entry in the magnitude relative to the entries in its row. Hence, We interchange both the columns and rows to find the largest entry in absolute value, i.e. we searching for largest entry un the matrix and for this reason this type of pivoting is also known as **maximal pivoting**. we start the

process by searching all the rows r to n for the largest entry in absolute value in each row, we denote this element by p_k:

$$p_k = \max\{|a_{kr}|, |a_{kr+1}|, \cdots, |a_{kn-1}|, |a_{kn}|\},\ k = r, r+1, \cdots, n. \quad (1.2)$$

Then, to locate the pivoting row, we need to compute

$$\frac{a_{kr}}{p_k} = \max\{|\frac{a_{rr}}{p_r}|, |\frac{a_{r+1r}}{p_{r+1}}|, \cdots, |\frac{a_{n-1r}}{p_{n-1}}|, |\frac{a_{nr}}{p_n}|\}. \quad (1.3)$$

Then, interchange the row r and k, except the case when $r = k$.

Example 7. *Write the following linear system in the augmented form and then solve it by using Gauss elimination method with trivial pivoting.*

$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= 12 \\ 2x_1 + x_2 + x_3 + x_4 &= 10 \\ -3x_1 - x_2 + 4x_3 + x_4 &= 2 \\ x_1 + x_2 - x_3 + 3x_4 &= 6 \end{aligned}$$

.

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 2 & 1 & 1 & 1 & 10 \\ -3 & -1 & 4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 6 \end{array}\right]$$

The first row is the pivotal row, so the pivotal element is $a_{11} = 1$ and is used to eliminate the first column below the diagonal. We will denote by m_{r1} to the multiples of the row 1 subtracted from row r for $r = 2, 3, 4$. Multiplying the first row by $m_{21} = -2$ and add it to the second row to have

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ -3 & -1 & 4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 6 \end{array}\right].$$

Now, multiply the first row by $m_{31} = 3$ and add it to the third row to obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 5 & 1 & 13 & 38 \\ 1 & 1 & -1 & 3 & 6 \end{array}\right].$$

Multiplying the first row by $m_{41} = -1$ and adding it to the fourth row yields

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 5 & 1 & 13 & 38 \\ 0 & -1 & 0 & -1 & -6 \end{array}\right].$$

Now, the pivotal row is the second row and the pivotal element is $a_{22} = -3$. Multiply the second row by $m_{32} = \frac{5}{3}$ to have

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 0 & 6 & 4/3 & 44/3 \\ 0 & -1 & 0 & -1 & -6 \end{array}\right].$$

Multiply the the second row by $m_{42} = \frac{-1}{3}$ and add it to the fourth row to obtain

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 0 & 6 & 4/3 & 44/3 \\ 0 & 0 & -1 & 4/3 & -4/3 \end{array}\right].$$

Now, the pivotal row is the third row and the third element is $a_{33} = 6$. Finally, multiply the third row by $m_{43} = \frac{1}{6}$ to the fourth row to have

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 12 \\ 0 & -3 & 3 & -7 & -14 \\ 0 & 0 & 6 & 4/3 & 44/3 \\ 0 & 0 & 0 & 14/9 & 10/9 \end{array}\right].$$

Now, note that the coefficient matrix is transformed into an upper triangular matrix and can be solved by backward substitution method. Firstly, we from the last row we compute

$$x_4 = \frac{10/9}{14/9} = \frac{5}{7}.$$

Use the third row to solve for x_3

$$x_3 = \frac{44/3 - 4/3(5/7)}{6} = \frac{288/21}{6} = \frac{16}{7}.$$

Now, solve the second equation for x_2

$$x_2 = \frac{-14 - 3x_3 + 7x_4}{-3} = \frac{-14 - 3(16/7) + 7(5/7)}{-3} = \frac{111}{21} = \frac{37}{7}.$$

Finally, solve the first equation for x_1

$$x_1 = 12 - 2x_2 + x_3 - 4x_4 = 12 - 2(37/7) + 16/7 - 4(5/7) = \frac{6}{7}.$$

Example 8. *Solve the following linear system using Gauss elimination method by using forward substitution technique*

$$\begin{aligned} x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\ 2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\ 4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\ -3x_1 + x_2 + 3x_3 + 2x_4 &= 6 \end{aligned}$$

.

Solution: We start our solution strategy by transforming this square system to equivalent lower-triangular system and then solve it by using forward substitution method. Write the system in augmented matrix form

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

$$\begin{array}{cc} \begin{array}{ccccc} a & b & c & d & e \end{array} & \\ \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{c} R_1 + 2R_2 \\ g \\ h \\ i \\ j \end{array} \end{array}$$

Note that now the pivotal row is the fourth row and the pivotal element is $a_{44} = 2$. Multiply the fourth row by the multiple $m_{14} = -2$ and it to the first row to have

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Multiply the fourth row by $m_{24} = \frac{-3}{2}$ and add it to the second row to obtain

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 13/2 & -3/2 & -1/2 & 0 & 19 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Now multiply the fourth equation by $m_{34} = \frac{-1}{2}$ and add it to the third row to have

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 13/2 & -3/2 & -1/2 & 0 & 19 \\ 11/2 & 3/2 & 1/2 & 0 & 17 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

The pivotal row now is the third row and the pivotal element is $a_{33} = 1/2$. Add the third row to the second row (i.e. multiply it by $m_{23} = 1$) to get

$$\left[\begin{array}{cccc|c} 7 & 0 & -5 & 0 & 1 \\ 12 & 0 & 0 & 0 & 36 \\ 11/2 & 3/2 & 1/2 & 0 & 17 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Now

$$\left[\begin{array}{cccc|c} 12 & 0 & 0 & 0 & 36 \\ 11/2 & 3/2 & 1/2 & 0 & 17 \\ 7 & 0 & -5 & 0 & 1 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

The pivotal row (third row) is used to eliminate elements in the second row and the pivotal element is $a_{33} = -5$. Multiply the third row by $m_{23} = \frac{1}{10}$ to have

$$\left[\begin{array}{cccc|c} 12 & 0 & 0 & 0 & 36 \\ 31/5 & 3/2 & 0 & 0 & 171/10 \\ 7 & 0 & -5 & 0 & 1 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right].$$

Now, use forward substitution to solve the lower-triangular matrix. solve the first equation for x_1

$$x_1 = \frac{36}{12} = 3.$$

Use the equation to find x_2

$$x_2 = \frac{171/10 - (31/5)3}{3/2} = -1.$$

Now, solve the third equation for x_3

$$x_3 = \frac{1 - 7(3)}{-5} = 4.$$

Finally, solve the fourth equation for x_4

$$x_4 = \frac{6 - (-3)(3) - 1(-1) - 3(4)}{2} = 2.$$

1.2.4 Gauss-Jordan Elimination Method

In this method instead of transforming the coefficient matrix into upper or lower triangular system, we transform the coefficient matrix into diagonal (in particular identity) matrix using elementary row operations.

Example 9. *Solve the following linear system using Gauss-Jordan elimination method*

$$\begin{aligned} 3x_1 + 4x_2 + 3x_3 &= 10 \\ x_1 + 5x_2 - x_3 &= 7 \\ 6x_1 + 3x_2 + 7x_3 &= 15 \end{aligned}$$

.

Solution: Express the system in augmented matrix form

$$\left[\begin{array}{ccc|c} 3 & 4 & 3 & 10 \\ 1 & 5 & -1 & 7 \\ 6 & 3 & 7 & 15 \end{array}\right].$$

The pivot row is the first row and the pivot element is $a_{11} = 3$. Multiply it by $m_{11} = 1/3$ to get

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 1 & 10/3 \\ 1 & 5 & -1 & 7 \\ 6 & 3 & 7 & 15 \end{array}\right].$$

Subtract the second equation from the first (i.e. multiply it by $m_{21} = -1$) and multiply the fist equation by $m_{31} = -6$ and add it to the third equation to have

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 1 & 10/3 \\ 0 & -11/3 & 2 & -11/3 \\ 0 & -5 & 1 & -5 \end{array}\right].$$

Now, the pivot row is the second row and the pivot element $a_{22} = -11/3$. Multiply it by $m_{22} = -3/11$ to have

$$\left[\begin{array}{ccc|c} 1 & 4/3 & 1 & 10/3 \\ 0 & 1 & -6/11 & 1 \\ 0 & -5 & 1 & -5 \end{array}\right].$$

Multiply the first and third rows by $m_{12} = -4/3$ and $m_{32} = 5$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 19/11 & 2 \\ 0 & 1 & -6/11 & 1 \\ 0 & 0 & -19/11 & 0 \end{array}\right].$$

The pivot element now is third row and the pivot element is $a_{33} = -19/11$. Multiply it by $m_{33} = -11/19$ to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 19/11 & 2 \\ 0 & 1 & -6/11 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Finally, multiply the third row by $m_{13} = -19/11$ and $m_{23} = 6/11$ and add it to the first and second rows to have

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Hence, we have $x_1 = 2$, $x_2 = 1$ and $x_3 = 0$.

Example 10. *Solve the following linear system using Gauss-Jordan elimination method*

$$\begin{aligned} -2x_1 + x_2 + 5x_3 &= 15 \\ 4x_1 - 8x_2 + x_3 &= -21 \\ 4x_1 - x_2 + x_3 &= 7 \end{aligned}$$

.

Solution: Write the system in augmented matrix form

$$\left[\begin{array}{ccc|c} -2 & 1 & 5 & 15 \\ 4 & -8 & 1 & -21 \\ 4 & -1 & 1 & 7 \end{array}\right].$$

Multiply the first row by $m_{21} = m_{31} = -2$ and it to the second and third rows respectively, to obtain

$$\left[\begin{array}{ccc|c} -2 & 1 & 5 & 15 \\ 0 & -6 & 11 & 9 \\ 0 & 1 & 11 & 37 \end{array}\right].$$

Now, multiply the second row by $m_{12} = m_{32} = \frac{1}{6}$ and it to the first and third rows respectively, to have

$$\left[\begin{array}{ccc|c} -2 & 0 & 41/6 & 33/2 \\ 0 & -6 & 11 & 9 \\ 0 & 0 & 77/6 & 77/2 \end{array}\right].$$

Finally, multiply the third row by $m_{13} = \frac{-41}{77}$ and $m_{32} = \frac{-6}{7}$ and it to the first and third rows respectively, to obtain

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & -4 \\ 0 & -6 & 0 & -24 \\ 0 & 0 & 77/6 & 77/2 \end{array}\right],$$

implies that

$$x_1 = \frac{-4}{-2} = 2, \quad x_2 = \frac{-24}{-6} = 4 \quad \text{and} \quad x_3 = \frac{77/2}{77/6} = 3.$$

1.3 LU and Cholesky Factorisations

In this section we will discuss the triangular factorisations of matrices.

Definition 4 (Positive Definite Matrix)**.** *Let $A_{n\times n}$ be symmetric real matrix and $\mathbf{x} \in \mathbb{R}^n$ a nonzero vector. Then, A is said to be **positive definite matrix** if $A = A^T$ and $\mathbf{x}^T A\mathbf{x} > 0$ for any $\mathbf{x}$.*

Remark 1. *Note that the matrix A is nonsingular by definition.*

Definition 5 (Triangular Factorisation)**.** *Assume that A is a nonsingular matrix. It said to be A has a **triangular factorisation** or **triangular decomposition** if it can be factorised as a product of unit lower-triangular matrix L and an upper triangular matrix U:*

$$A = LU.$$

or in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

*Note that since A is nonsingular matrix this implies that $u_{rr} \neq 0$ for all r and this is called **Doolittle factorisation**.*

Also, A can be expressed as a product of lower-triangular matrix L and unit upper triangular matrix U:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix},$$

*and this is called **Crout factorisation**.*

To solve the linear system $AX = B$ using LU factorisation, we do the following two steps:

1. Using forward substitution to solve the the lower-triangular linear system $LY = B$ for Y.
2. Using backward substitution to solve the upper-triangular linear system $UX = Y$ for X.

Direct LU Factorisation Using Gaussian Elimination Method

The matrix A can be factored directly using Gauss elimination method without any row interchanges. In this case the matrix A is expressed in terms of the identity matrix I follows $A = IA$. We perform the row operations on the matrix A on the right and the resulting matrix it will be the upper triangular matrix U. The multipliers are stored in their appropriate places in the identity matrix on the left which will be the lower triangular matrix L. All this information is summarised in the next theorem.

Theorem 1 (Direct LU Factorisation Without Row Interchanges). *Assume that the linear system $AX = B$ can be solved using Gaussian elimination with no row interchanges. Then, the coefficient matrix A can be factored as a product of a lower triangular matrix L and an upper triangular matrix U as follows:*

$$A = LU.$$

The matrix L has 1's on its main diagonal and the matrix has nonzero entries on its main diagonal. After constructing the matrices L and U then the linear system can be solved in the following two steps:

(1). Solve the lower triangular system $LY = B$ for Y using the forward substitution method.

(2). Solve the upper triangular system $UX = Y$ for X using the backward substitution method.

Proof. For proof, see any standard text on numerical analysis or numerical linear algebra. □

The following example explains this type of LU factorisation.

Example 11. *Find the LU factorisation of the following matrix using Gaussian elimination without row interchanges*

$$A = \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix}.$$

Solution. *Writing the matrix A in terms of the identity matrix as follows*

$$A = \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = IA$$

The first row is used to eliminate the elements under the main diagonal (subdiagonal elements) in the first column. The multipliers of the first row are $m_{21} = a_{21}/a_{11} = -2/2 = -1$ *and* $m_{31} = a_{31}/a_{11} = 1/2 = 0.5$, *respectively.*

$$\begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 3 & 6.5 \end{bmatrix}$$

Now, the second row is used to eliminate the entries below the main diagonal in the second column and the multiple of the second row is $m_{32} = a_{32}/a_{22} = 3/7$. *Hence, we have the following LU factorisation of A*

$$A = \begin{bmatrix} 2 & 4 & -1 \\ -2 & 3 & 1 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 3/7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 6.5 \end{bmatrix} = LU.$$

The LU Factorisation Without Using Gaussian Elimination Method

Example 12. *Solve the following linear system using LU (Doolittle) decomposition*

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 2 \\ x_1 + x_2 - x_3 &= -1 \\ -x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Solution: Express the system in matrix form

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 0 \end{array}\right].$$

Factor A as follows:

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Find the values of the entries of matrices L and U. From the first column we have

$$2 = 1u_{11} \implies u_{11} = 2,$$

and

$$1 = l_{21}u_{11} = l_{21}2 \implies l_{21} = 0.5,$$

finally

$$-1 = l_{31}u_{11} = l_{31}2 \implies l_{31} = -0.5.$$

In the second column, we have

$$-3 = 1u_{12} \implies u_{12} = -3,$$

and

$$1 = l_{21}u_{12} + 1u_{22} = -1.5 + u_{22} \implies u_{22} = 2.5,$$

so

$$1 = l_{31}u_{12} + l_{32}u_{22} = (-0.5)(-3) + l_{32}(2.5) \implies l_{32} = -0.2.$$

Finally, in the third column we have

$$1 = 1u_{13} \implies u_{13} = 1,$$

and

$$-1 = l_{21}u_{13} + 1u_{23} = 0.5 + u_{23} \implies u_{23} = -1.5,$$

finally,

$$-1 = l_{31}u_{13} + l_{32}u_{23} + 1u_{33} = -0.5(1) + (-0.2)(-1.5) + u_{33} \implies u_{33} = -0.8.$$

Now, we have the LU factorisation

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 2.5 & -1.5 \\ 0 & 0 & -0.8 \end{bmatrix} = LU.$$

Now, we have the following lower-triangular linear system $LY = B$ for Y

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

Write the system in augmented matrix form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0.5 & 1 & 0 & -1 \\ -0.5 & -0.2 & 1 & 0 \end{array}\right].$$

Solve this system by forward substitution to have

$$y_1 = 2, \quad y_2 = -1 - 0.5(y_1) = -1 - 0.5(2) = -2,$$

and

$$y_3 = 0 + 0.5(y_1) + 0.2(y_2) = 0.5(2) + 0.2(-2) = 0.6.$$

Now, we have the following upper-triangular linear system $UX = Y$

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 2.5 & -1.5 \\ 0 & 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0.6 \end{bmatrix}.$$

Express the system in augmented matrix form

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 2 \\ 0 & 2.5 & -1.5 & -2 \\ 0 & 0 & -0.8 & 0.6 \end{array}\right].$$

Finally, use the values of Y to solve the upper-triangular linear system $UX = Y$ by back substitution to have

$$x_3 = \frac{0.6}{-0.8} = \frac{-3}{4}, \quad x_2 = \frac{-2+1.5(x_3)}{2.5} = \frac{-2+1.5(-3/4)}{2.5} = -5/4,$$

and

$$x_1 = \frac{2+3(x_2)-1(x_3)}{2} = \frac{2+3(-5/4)-(-3/4)}{2} = -1/2.$$

Definition 6 (Cholesky Factorisation)**.** *Let A be a real, symmetric and positive definite matrix. Then, it can be* ***factored*** *or* ***decomposed*** *in a unique way $A = LL^T$, in which L is a lower-triangular matrix with a positive diagonal, and is termed* ***Cholesky factorisation****.*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}.$$

Exercises

Exercise 2. *Solve Example 7 Using Gauss elimination with forward substitution method. Compare the solution with solution of the same example.*

Exercise 3. *Solve Example 8 Using Gauss elimination with backward substitution method. Compare the solution with solution of the same example.*

Exercise 4. *Use Gauss elimination with backward substitution method and three-digit rounding arithmetic to solve the following linear system*

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 - 3x_3 &= -2 \\ x_1 + 5x_2 + 3x_3 &= 10 \end{aligned}$$

.

Exercise 5. *(a) Determine the LU factorisation for matrix A in the linear system $AX = B$, where*

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & -1 & 1 \\ -4 & 1 & -2 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

(b) Then use the factorisation to solve the system

$$\begin{aligned} -x_1 + x_2 - 2x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 1 \\ -4x_1 + x_2 - 2x_3 &= 4 \end{aligned}$$

.

Exercise 6. *Solve the following linear system using Gauss-Jordan elimination method*

$$\begin{aligned} -4x_1 - x_2 - 2x_3 &= -9 \\ -x_1 - x_2 + 3x_3 &= 9 \\ -2x_1 - 4x_2 + x_3 &= 5 \end{aligned}$$

.