Norms of Matrices and Vectors

Lecture Notes

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0.1 Norms of Matrices and Vectors

In error and convergence analyses we need a measure to determine the distance (difference) between the exact solution and approximate solution or to determine the differences between consecutive approximations.

Definition 1 (Vector Norm)**.** *A **vector norm** is a real-valued function $\|.\| : \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions:*

(i) $\|\mathbf{x}\| \geq 0$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

(ii) $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ *for all* $\alpha \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$.

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *(Triangle Inequality).*

Definition 2 (l_1 Vector Norm)**.** *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. *Then the* l_1 ***norm*** *for the vector* $\mathbf{x}$ *is defined by*

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

Definition 3 (Euclidean Vector Norm)**.** *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. *Then the **Euclidean norm** (l_2 **norm**) for the vector* $\mathbf{x}$ *is defined by*

$$\|\mathbf{x}\|_2 = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}.$$

Definition 4 (Maximum Vector Norm)**.** *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. *Then the **maximum norm** (l_∞ **norm**) for the vector* $\mathbf{x}$ *is defined by*

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Remark 1. *Note that when $n = 1$ both norms reduce to the absolute value function of real numbers.*

Example 1. *Determine the l_1 norm, l_2 norm and l_∞ norm of the vector* $\mathbf{x} = (1, 0, -1, 2, 3)'$.

Solution: The required norms of vector $\mathbf{x} = (1, 0, -1, 2, 3)'$ in $\mathbb{R}^5$ are:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{5} |x_i| = |x_1| + |x_2| + |x_3| + |x_4| + |x_5| = |1| + |0| + |-1| + |2| + |3| = 7,$$

$$\begin{aligned}\|\mathbf{x}\|_2 &= \Big(\sum_{i=1}^{5} x_i^2\Big)^{1/2} = \Big(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\Big)^{1/2}\\ &= \Big((1)^2+(0)^2+(-1)^2+(2)^2+(3)^2\Big)^{1/2} = \Big(15\Big)^{1/2},\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{x}\|_\infty &= \max_{1\le i\le 5}|x_i| = \max\{|x_1|,|x_2|,|x_3|,|x_4|,|x_5|\}\\ &= \max\{|1|,|0|,|-1|,|2|,|3|\} = 3.\end{aligned}$$

Definition 5 (Matrix Norm)**.** *A* ***matrix norm*** *is a real-valued function* $\|.\| : \mathbb{R}^{n\times m} \to \mathbb{R}$ *satisfies the following conditions:*

(i) $\|A\| \geq 0$ *for all* $A \in \mathbb{R}^{n\times m}$.

(ii) $\|A\| = 0$ *if and only if* $A = \mathbf{0}$ *for all* $A \in \mathbb{R}^{n\times m}$.

(iii) $\|\alpha A\| = |\alpha|\|A\|$ *for all* $\alpha \in \mathbb{R}$ *and* $A \in \mathbb{R}^{n\times m}$.

(iv) $\|A+B\| \leq \|A\| + \|B\|$ *for all* $A, B \in \mathbb{R}^{n\times m}$ *(Triangle Inequality).*

If matrix norm is related to a vector norm, then we have two additional properties:

(v) $\|AB\| \leq \|A\|\|B\|$ *for all* $A, B \in \mathbb{R}^{n\times m}$.

(vi) $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$ *for all* $A \in \mathbb{R}^{n\times m}$ *and* $\mathbf{x} \in \mathbb{R}^n$.

We give here some equivalent definitions of the matrix norm particularly when matrix norm is related to the vector norm.

Definition 6 (Subordinate Matrix Norm)**.** *Let* A *is a* $n \times n$ *matrix and* $\mathbf{x} \in \mathbb{R}^n$, *then the* ***subordinate matrix norm*** *is defined by*

$$\|A\| = \sup\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n \, and \, \|\mathbf{x}\| = 1\}.$$

or, alternatively

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Definition 7 (Natural Matrix Norm)**.** *Let A is a $n \times n$ matrix and for any $\mathbf{z} \neq \mathbf{0}$, and $\mathbf{x} = \frac{\mathbf{z}}{\|\mathbf{z}\|}$ is the unit vector. Then the **natural / reduced matrix norm** is defined by*

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \max_{\mathbf{z}\neq 0} \left\| A\Big(\frac{\mathbf{z}}{\|\mathbf{z}\|}\Big) \right\| = \max_{\mathbf{z}\neq 0} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|},$$

or, alternatively

$$\|A\| = \max_{\mathbf{z}\neq 0} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|}.$$

Definition 8 (l_1 Matrix Norm)**.** *Let A is a $n \times n$ matrix and $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. Then the l_1 **matrix norm** is defined by*

$$\|A\|_1 = \max_{\|\mathbf{x}\|_1=1} \|A\mathbf{x}\|_1 = \max_{1\leq i\leq n} \sum_{i=1}^{n} |a_{ij}|.$$

Definition 9 (Spectral Matrix Norm)**.** *Let A is a $n \times n$ matrix and $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. Then the **spectral / l_2-matrix norm** is defined by*

$$\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 = \max_{1\leq i\leq n} \sqrt{|\sigma_{\max}|},$$

*where σ_i are the eigenvalues of $A^T A$, which are called the **singular values** of A and the largest eigenvalue in absolute value ($|\sigma_{\max}|$) is called the **spectral radius** of A.*

Definition 10 (l_∞ Matrix Norm)**.** *Let A is a $n \times n$ matrix and $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$. Then the l_∞ **(maximum)matrix norm** is defined by*

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|A\mathbf{x}\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$$

Remark 2. *Note that $\|I\| = 1$.*

Example 2. *Determine $\|A\|_\infty$ for the matrix*

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & 3 \\ -1 & 6 & -4 \end{bmatrix}.$$

Solution: For $i = 1$, we have

$$\sum_{j=1}^{3} |a_{1j}| = |a_{11}| + |a_{12}| + |a_{13}| = |1| + |-1| + |2| = 4,$$

and for $i = 2$, we obtain

$$\sum_{j=1}^{3} |a_{2j}| = |a_{21}| + |a_{22}| + |a_{23}| = |0| + |5| + |3| = 8,$$

for $i = 3$, we get

$$\sum_{j=1}^{3} |a_{3j}| = |a_{31}| + |a_{32}| + |a_{33}| = |-1| + |6| + |-4| = 11.$$

Consequently,

$$\|A\|_{\infty} = \max_{1 \le i \le 3} \sum_{j=1}^{3} |a_{ij}| = \max\{4, 8, 11\} = 11.$$