Numerical Integration

Lecture Notes

Mohammad Sabawi

Department of Mathematics
College of Education for Women
Tikrit University

Email: mohammad.sabawi@tu.edu.iq

01 October 2021

Contents

Chapter 1

Numerical Integration

1.1 Introduction

In solving daily life problems, we sometimes encountered with integrations problems whose integrals cannot be computed analytically. In these scenarios we resort to numerical integration to integrate these problems numerically by using approximating methods. In these lecture notes, we focus on Newton-Cotes formulas of integration.

1.2 Newton-Cotes Formulas of Integration

Newton-Cotes quadrature formulas are numerical formulas used to approximate the definite integral $\int_a^b f(x)\,dx$. In these formulas the function of integration or the **integrand** is replaced by an interpolating polynomial. These integration rules are said to be **closed** if they include the function values at the end of the integration interval. Otherwise, they are called **open**.

1.2.1 Closed Newton-Cotes Integration Rules

The general structure of these formulas is as follows: Let $a = x_0, b = x_n$ and the step size $h = \frac{b-a}{n}$, then the internal integration points can be defined by $x_i = x_0 + ih, i = 1, \cdots, n$, where $f_i = f(x_i), i = 0, \cdots, n$. Here, we consider some of these closed rules such as:

(a) **Trapezoid Rule**:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{1}{2}h[f_0 + f_1].$$

(b) **Simpson's $\frac{1}{3}$ Rule**:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{1}{3}h[f_0 + 4f_1 + f_2].$$

(c) **Simpson's $\frac{3}{8}$ Rule**:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{8}h[f_0 + 3f_1 + 3f_2 + f_3].$$

(d) **Boole's Rule**:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2}{45}h[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4].$$

(e) **Six-Point Newton-Cotes Closed Rule**:

$$\int_{x_0}^{x_5} f(x)\,dx = \frac{5}{288}h[19f_0 + 75f_1 + 50f_2 + 50f_3 + 75f_4 + 19f_5].$$

Example 1. *Consider the function $f(x) = e^{x^2}$, use the above-mentioned quadrature rules to approximate the integral $\int_0^{0.6} e^{x^2}\,dx$.*

Solution: Let us divide the interval of integration $I = [a, b] = [0, 0.6]$ to six equal subintervals (i.e. $n = 6$). Hence, $h = (b - a)/n = (0.6 - 0)/6 = 0.1$. The function values at the end points are:

$$f_0 = f(x_0) = e^{x_0^2} = e^{(0)^2} = 1,$$

and

$$f_6 = f(x_6) = e^{x_6^2} = e^{(0.6)^2} = 1.43332941.$$

Now, we compute the points of integration $x_i = x_0 + ih, i = 1, \cdots, 6$.

So, when $i = 1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1,\ f_1 = f(x_1) = e^{x_1^2} = e^{(0.1)^2} = 1.01005017,$$

when $i = 2$

$$x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2,\ f_2 = f(x_2) = e^{x_2^2} = e^{(0.2)^2} = 1.04081078,$$

when $i = 3$

$$x_3 = x_0 + 3h = 0 + 3(0.1) = 0.3,\ f_3 = f(x_3) = e^{x_3^2} = e^{(0.3)^2} = 1.09417428,$$

when $i = 4$

$$x_4 = x_0 + 4h = 0 + 4(0.1) = 0.4,\ f_4 = f(x_4) = e^{x_4^2} = e^{(0.4)^2} = 1.17351087,$$

when $i = 4$

$$x_5 = x_0 + 5h = 0 + 5(0.1) = 0.5,\ f_5 = f(x_5) = e^{x_5^2} = e^{(0.5)^2} = 1.28402542,$$

The computations are summarised in the table below:

x	$f(x)$
0	1
0.1	1.01005017
0.2	1.04081078
0.3	1.09417428
0.4	1.17351087
0.5	1.28402542
0.6	1.43332941

(a) **Trapezoid Rule**: The interval of integration is $I = [a, b] = [0, 0.6]$, so, $n = 1$. Hence, $h = (b - a)/n = (0.6 - 0)/1 = 0.6$. The function values at the end points $x_0 = 0$ and $x_1 = 0.6$ are:

$$f_0 = f(x_0) = e^{x_0^2} = e^{(0)^2} = 1,$$

and

$$f_1 = f(x_1) = e^{x_6^2} = e^{(0.6)^2} = 1.43332941.$$

Consequently, we get

$$\int_{x_0}^{x_1} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = \frac{1}{2}h[f_0+f_1] = \frac{1}{2}(0.6)[1+1.43332941] = 0.72999882.$$

(b) **Simpson's $\frac{1}{3}$ Rule**: We divide the integration interval $I = [a, b] = [0, 0.6]$ to two equal subintervals (i.e. $n = 2$). Hence, $h = (b - a)/n = (0.6 - 0)/2 = 0.3$. The integration nodes are $x_0 = 0$, $x_1 = x_0 + h = 0 + (1)(0.3) = 0.3$, and $x_2 = 0.6$. The functional values at the integration nodes are:

$$f_0 = 1,\ f_1 = f(x_1) = e^{x_1^2} = e^{(0.3)^2} = 1.09417428,\ f_2 = 1.43332941.$$

$$\int_{x_0}^{x_2} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = \frac{1}{3}h[f_0 + 4f_1 + f_2] = \frac{1}{3}(0.3)[1 + 4(1.09417428) + 1.43332941] = 0.68100265.$$

(c) **Simpson's $\frac{3}{8}$ Rule**: In this rule, we divide the integration interval $I = [0, 0.6]$ to three equal subintervals (i.e. $n = 3$). Hence, $h = (b - a)/n = (0.6 - 0)/3 = 0.2$. The integration nodes are $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.2) = 0.2$, $x_2 = x_0 + 2h = 0 + 2(0.2) = 0.4$ and $x_3 = 0.6$. The functional values at the integration nodes are:

$$f_0 = 1,\ f_1 = 1.04081078,\ f_2 = 1.17351087,\ f_3 = 1.43332941.$$

$$\int_{x_0}^{x_3} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = \frac{3}{8}h[f_0 + 3f_1 + 3f_2 + f_3] = \frac{3}{8}(0.2)[1 + 3(1.04081078) + 3(1.17351087) + 1.43332941] = 0.68072208.$$

(d) **Boole's Rule**: In Boole's rule, we divide the integration interval $I = [0, 0.6]$ to four equal subintervals, $n = 4$. So, $h = (b-a)/n = (0.6-0)/4 = 0.15$. The integration nodes are $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.15) = 0.15$, $x_2 = x_0 + 2h = 0 + 2(0.15) = 0.3$, $x_3 = x_0 + 3h = 0 + 3(0.15) = 0.45$ and $x_4 = 0.6$. The functional values at the integration nodes are:

$$f_0 = 1,\ f_1 = 1.02275503,\ f_2 = 1.09417428,\ f_3 = 1.22446006,\ f_4 = 1.43332941.$$

$$\begin{aligned}\int_{x_0}^{x_4} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx &= \frac{2}{45}h[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4] \\ &= \frac{2}{45}(0.15)[7(1) + 32(1.02275503) + 12(1.09417428) \\ &\quad +32(1.22446006) + 7(1.43332941)] = 0.68049520.\end{aligned}$$

(e) **Six-Point Newton-Cotes Closed Rule**: We divide the integration interval $I = [0, 0.6]$ to five equal subintervals, $n = 5$. So, $h = (b - a)/n = (0.6 - 0)/5 = 0.12$. The integration nodes are $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.12) = 0.12$, $x_2 = x_0 + 2h = 0 + 2(0.12) = 0.24$, $x_3 = x_0 + 3h = 0 + 3(0.12) = 0.36$, $x_4 = x_0 + 4h = 0 + 4(0.12) = 0.48$ and $x_5 = 0.6$. The functional values at the integration nodes are:

$$f_0 = 1,\ f_1 = 1.01450418,\ f_2 = 1.05929119,\ f_3 = 1.13837294,\ f_4 = 1.25910355,\ f_5 = 1.43332941.$$

$$\begin{aligned}\int_{x_0}^{x_5} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx &= \frac{5}{288}h[19f_0 + 75f_1 + 50f_2 + 50f_3 + 75f_4 + 19f_5] \\ &= \frac{5}{288}(0.12)[19(1) + 75(1.01450418) + 50(1.05929119) \\ &\quad +50(1.13837294) + 75(1.25910355) + 19(1.43332941)] = 0.68049384.\end{aligned}$$

1.2.2 Open Newton-Cotes Integration Rules

The general structure of these formulas is the same as the general structure of the closed rules except that the two end points $x_0 = a, x_n = b$ and their functional values $f(x_0) = f(a), f(x_n) = f(b)$ are not included in the integrations formulas. We consider the following open rules:

(a) **Midpoint Rule**:

$$\int_{x_0}^{x_2} f(x)\,dx = 2hf_1.$$

(b) **Two-Point Newton-Cotes Open Rule**:

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3}{2}h[f_1 + f_2].$$

(c) **Three-Point Newton-Cotes Open Rule**:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{4}{3}h[2f_1 - f_2 + 2f_3].$$

(d) **Four-Point Newton-Cotes Open Rule**:

$$\int_{x_0}^{x_5} f(x)\,dx = \frac{5}{24}h[11f_1 + f_2 + f_3 + 11f_4].$$

(e) **Five-Point Newton-Cotes Open Rule**:

$$\int_{x_0}^{x_6} f(x)\,dx = \frac{6}{20}h[11f_1 - 14f_2 + 26f_3 - 14f_4 + 11f_5].$$

Example 2. *Redo Example 1 use the above-mentioned quadrature open rules to approximate the integral* $\int_0^{0.6} e^{x^2}\,dx$.

(a) **Midpoint Rule**: We divide the integration interval $I = [0, 0.6]$ to two equal subintervals, $n = 2$. So, $h = (b - a)/n = (0.6 - 0)/2 = 0.3$, so, $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.3) = 0.3$ and $x_2 = 0.6$. The function values at the integration points are:

$$f_0 = 1,\ f_1 = 1.09417428,\ f_2 = 1.43332941.$$

$$\int_{x_0}^{x_2} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = 2hf_1 = 2(0.3)(1.09417428) = 0.65650457.$$

(b) **Two-Point Newton-Cotes Open Rule**: Now, we divide the interval of integration $I = [0, 0.6]$ to three equal subintervals, i.e. $n = 3$. So, $h = (b - a)/n = (0.6 - 0)/3 = 0.2$, so, $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.2) = 0.2$, $x_2 = x_0 + (2)h = 0 + (2)(0.2) = 0.4$ and $x_3 = 0.6$. The functional values at the integration nodes are:

$$f_0 = 1,\ f_1 = 1.04081078,\ f_2 = 1.17351087,\ f_3 = 1.43332941.$$

$$\int_{x_0}^{x_3} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = \frac{3}{2}h[f_1 + f_2] = \frac{3}{2}(0.2)[1.04081078 + 1.17351087] = 0.66429650.$$

(c) **Three-Point Newton-Cotes Open Rule**: In this rule, the interval of integration $I = [0, 0.6]$ is divided to four equal subintervals, i.e. $n = 4$. So, $h = (b - a)/n = (0.6 - 0)/4 = 0.15$, so, $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.15) = 0.15$, $x_2 = x_0 + (2)h = 0 + (2)(0.15) = 0.3$, $x_3 = x_0 + (3)h = 0 + (3)(0.15) = 0.45$ and $x_4 = 0.6$. The values of the function at the integration nodes are:

$$f_0 = 1,\ f_1 = 1.02275503,\ f_2 = 1.09417428,\ f_3 = 1.22446006,\ f_4 = 1.43332941.$$

$$\int_{x_0}^{x_4} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = \frac{4}{3}h[2f_1 - f_2 + 2f_3] = \frac{4}{3}(0.15)[2(1.02275503) - 1.09417428 + 2(1.22446006)] = 0.68005118.$$

(d) **Four-Point Newton-Cotes Open Rule**: We divide the interval of integration $I = [0, 0.6]$ to five equal subintervals, i.e. $n = 5$. So, $h = (b - a)/n = (0.6 - 0)/5 = 0.12$, hence, $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.12) = 0.12$, $x_2 = x_0 + (2)h = 0 + (2)(0.12) = 0.24$, $x_3 = x_0 + (3)h = 0 + (3)(0.12) = 0.36$, $x_4 = x_0 + (4)h = 0 + (4)(0.12) = 0.48$ and $x_5 = 0.6$. The function values at the integration nodes are:

$$f_0 = 1,\ f_1 = 1.01450418,\ f_2 = 1.05929119,\ f_3 = 1.13837294,\ f_4 = 1.25910355,\ f_5 = 1.43332941.$$

$$\int_{x_0}^{x_5} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = \frac{5}{24}h[11f_1 + f_2 + f_3 + 11f_4] = \frac{5}{24}(0.12)[11(1.01450418) + 1.05929119 + 1.13837294 + 11(1.25910355)] = 0.68018373.$$

(e) **Five-Point Newton-Cotes Open Rule**: In this rule, the interval of integration $I = [0, 0.6]$ is divided to six equal subintervals, i.e. $n = 6$. So, $h = (b - a)/n = (0.6 - 0)/6 = 0.1$, so, $x_0 = 0$, $x_1 = x_0 + (1)h = 0 + (1)(0.1) = 0.1$, $x_2 = x_0 + (2)h = 0 + (2)(0.1) = 0.2$, $x_3 = x_0 + (3)h = 0 + (3)(0.1) = 0.3$, $x_4 = x_0 + (4)h = 0 + (4)(0.1) = 0.4$, $x_5 = x_0 + (5)h = 0 + (5)(0.5) = 0.4$ and $x_6 = 0.6$. The values of the function at the integration nodes are:

$$f_0 = 1,\ f_1 = 1.01005017,\ f_2 = 1.04081078,\ f_3 = 1.09417428, f_4 = 1.17351087,\\ f_5 = 1.28402542,\ f_6 = 1.43332941.$$

$$\int_{x_0}^{x_6} f(x)\,dx = \int_0^{0.6} e^{x^2}\,dx = \frac{6}{20}h[11f_1 - 14f_2 + 26f_3 - 14f_4 + 11f_5]$$
$$= \frac{6}{20}(0.1)[11(1.01005017) - 14(1.04081078) + 26(1.09417428) -\\ 14(1.17351087) + 11(1.28402542)] = 0.68048579.$$