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عنوان المحاضرة : الخواص التبولوجية والوراثية في الفضاء التبولوجية

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Proposition (45)

Suppose that (M,τ) $and(N,\sigma)$ are homeomorphic spaces, if $(M,\tau)$is T_2_space, then (N,σ) is T_2_space.

Proof.

Let $f:(M,\tau) \to (N,\sigma)$ be $Home$ and if $u \neq v \in M$, then $f(u), f\ (v) \in N$and $f(u) \neq\ f\ (v)$.

Since $((M,\tau)$is T_2_space, we get $\exists\ H\ ,D$ are two disjoint $open$ sets of (M,τ)such that $f\ (u) \in H, f\ (v) \in D$.

But, $u \in f^{-1}(H)$, $v \in f^{-1}(D)$ and $f^{-1}(H) \cap f^{-1}(D) = f^{-1}(H \cap D) = f^{-1}(\emptyset) = \emptyset$. Hence (N,σ) is T_2_space. ■

Theorem (46)

Every compact and T_2_space is regular.

Proof.

Let (M,τ)be compact T_2_space, let $x \in M$ and F be subset of M, $\ni x \notin F$. Since (M,τ) is N_P_compact. we get F is compact. There four $\exists\ H, D$ are two $open$ sets $\ni x \in H$ and $F \subseteq D$, $H \cap D = \emptyset$. Then $(M,\tau)$is regular. ■

Definition (47)

The $(M,\tau)$is T_3_space if M is regular and T_1_space.

Corollary (48)

Every compact and T_2_space is T_3_space.

Proof.

Since T_2_space is T_1_space , by theorem (46). Every compact and T_2_space is regular. We have (M,τ) is T_3_space. ■

Proposition (49)

Every T_3_space is T_2_space.

Proof.

Let (M,τ)be T_3_space (regular and T_1_space) and $x,y \in M, x \neq y$.

Since $(M,\tau)$is T_1_space, then $\{x\},\{y\}$ is closed sets such that $x \notin \{y\}$.

Since $(M,\tau)$is $regular,\ then\ \exists\ H,D \in \tau,\ H \cap D = \emptyset, (x \in H \wedge \{y\} \subseteq D) \rightarrow x \in H \wedge y \in D$. Then $(M,\tau)$is T_2_space. ■

Theorem (50)

A compact and T_2_space is normal.

Proof.

Suppose that (M,τ) is compact T_2_space and if, $H,D \subseteq M$ are disjoint closed sets, became H,D are compact sets in M.

Let $x \in H$ be arbitrary and $H \cap D = \emptyset$, then $x \notin D$. By T_2_space, $\exists$ disjoint open sets $Z_x, I \in\ \tau \ni x \in Z_x, D \subseteq \tau$.

The family $\{Z_x : x \in\ H\}$, so$\{Z_{x_i}: i = 1,2,\dots,n\}$. Then $H \subseteq \bigcup_{i=1}^{n} Z_{x_i}$, we get $H \subseteq Z \rightarrow\ \bigcup_{i=1}^{n} Z_{x_i} = Z$. Furthermore, $D \subseteq I_i$ for $1 \leq i \leq n$, then $D \subseteq \bigcap_{i=1}^{n} I_i$, we get $D \subseteq I \rightarrow\ \bigcap_{i=1}^{n} I_i = I$.

Since $I_i \bigcap_1^n Z_{x_r} = \emptyset$ for $1 \leq i \leq n$,

$Z_{x_r} \bigcap_1^n I = Z_{x_r} \cap (\bigcap_{i=1}^{n} I_i)$= ∅for1 $\leq r \leq n,\ I \cap Z = I \cap \left(\bigcup_{i=1}^{n} Z_{x_i}\right) = \emptyset$

So, $H \subseteq Z, D \subseteq I$. Then (M,τ) is normal. ■

Definition (51)

The (M,τ) is T_4_space if (M,τ) is normal and T_1_space.

Corollary (52)

Every compact and T_2_space is T_4_space.

Proof.

Since T_2_space is T_1_space , by theorem above Every compact and T_2_space is normal. We have (M,τ) is T_4_space. ■

Theorem (53)

Every T_4_space is T_3_space.

Proof.

If $(M,\tau)$is T_4_space (N_P_normal and T_1_space) and let $x \neq y \in M$.
Since $(M,\tau)$is T_1_space $\rightarrow \{x\} and \{y\}$ is closed sets $\rightarrow x \notin \{y\}$, $y \notin \{x\}$ and $(M,\tau)$is $normal \rightarrow \exists$ $H, D \in \tau$, $H \cap D = \emptyset$,($\{x\} \subseteq$ H $\wedge \{y\} \subseteq D$)→ $x \in$ H $\wedge \{y\} \subseteq D$, that is the condition of the regular.
Then (M,τ) is T_3_space. ■

Remark (54)

Every regular and compact space is_normal.

By adding some conditions to the function, we get the following theorems.

Theorem (55)

Let $\mathfrak{f} : (M,\tau) \rightarrow (N,\sigma)$ be a bijective $\mathcal{P}$_open map and $\mathbb{X}$ is T_i_ spaces, then $\mathbb{Y}$ is $T_{\mathcal{P}i}$_spaces, where i =0,1,2.

Proof. We prove the case i =2.

Let $\mathcal{v}_2, \mathcal{u}_2$ be two points in $\mathbb{Y}$ and $\mathcal{v}_2 \neq \mathcal{u}_2$. Since $\mathfrak{f}$ is bijective , then $\exists \mathcal{v}_1, \mathcal{u}_1 \in \mathbb{X}$ and $\mathfrak{f}(\mathcal{v}_1) = \mathcal{v}_2, \mathfrak{f}(\mathcal{u}_1) = \mathcal{u}_2$. But , $\mathbb{X}$ is T_2_ spaces, then $\exists$ two disjoint open sets $\mathbb{H}, \mathbb{D} \in \mathbb{X}$,whenerver $\mathcal{v}_1 \in \mathbb{H}, \mathcal{u}_1 \in \mathbb{D}$. Then $\mathfrak{f}(\mathbb{H}), \mathfrak{f}(\mathbb{D})$ are $\mathcal{P}$_open sets in $\mathbb{Y}$ (because every $\mathcal{P}$_open is semi$_{\mathcal{P}}$O., and $\mathfrak{f}$ is $\mathcal{P}$_open), we get $\mathcal{v}_2 \in \mathfrak{f}(\mathbb{H}), \mathcal{u}_2 \in \mathfrak{f}(\mathbb{D})$ and $\mathfrak{f}(\mathbb{H}) \cap \mathfrak{f}(\mathbb{D}) = \emptyset$. Hence $\mathbb{Y}$ is $T_{\mathcal{P}2}$- space. ■

<u>Theorem (56)</u>

Let $\mathfrak{f} : (M, \tau) \rightarrow (N, \sigma)$ be injective Con$_{\mathcal{P}}$ map and $\mathbb{Y}$ is T_i_space.

Then $\mathbb{X}$ is $T_{\mathcal{P}i}$_ spaces, where i =0,1,2.

Proof. We prove the case i =1.

Since $\mathbb{Y}$ is T_1_ spaces and $\mathcal{v}, \mathcal{u}$ of $\mathbb{X} \ni \mathcal{v} \neq \mathcal{u}$, there exist two disjoint $\mathcal{P}$_open sets $\mathbb{H}, \mathbb{D} \in \mathbb{Y}$ such that $\mathfrak{f}(\mathcal{v}) \in \mathbb{H}, \mathfrak{f}(\mathcal{u}) \in \mathbb{D}, \mathfrak{f}(\mathcal{v}) \neq \mathfrak{f}(\mathcal{u})$. Since $\mathfrak{f}$ is $\mathcal{P}$_continuous, then $\mathfrak{f}^{-1}(\mathbb{H})$ and $\mathfrak{f}^{-1}(\mathbb{D})$ are $\mathcal{P}$_open sets of $\mathbb{X}$, we get $\mathcal{v} \in \mathfrak{f}^{-1}(\mathbb{H}), \mathcal{u} \in \mathfrak{f}^{-1}(\mathbb{D})$. Hence $\mathbb{X}$ is $T_{\mathcal{P}1}$- space. ■

<u>Theorem (57)</u>

If $\mathfrak{f} : (M, \tau) \rightarrow (N, \sigma)$ is injective Con$_{\mathcal{P}}$ map and $\mathbb{Y}$ is $T_{\mathcal{P}i}$_spaces, then $\mathbb{X}$ is $T_{\mathcal{P}i}$_spaces, where i=0,1,2.

Proof.

We prove the case i =2.

Suppose that $\mathcal{v}, \mathcal{u}$ of $\mathbb{X}$ and $\mathcal{v} \neq \mathcal{u}$. Since $\mathfrak{f}$ is one to one, then $\mathfrak{f}(\mathcal{v}) \neq \mathfrak{f}(\mathcal{u})$ in $\mathbb{Y}$. But $\mathbb{Y}$ is $T_{\mathcal{P}2}$_space, then $\exists$ two disjoint $\mathcal{P}$_open sets $\mathbb{H}, \mathbb{D} \in \mathbb{Y}$, whenever,$\mathfrak{f}(\mathcal{v}) \in \mathbb{H}, \mathfrak{f}(\mathcal{u}) \in \mathbb{D}$.Then $\mathfrak{f}^{-1}(\mathbb{H}), \mathfrak{f}^{-1}(\mathbb{D})$ are $\mathcal{P}$_open,we get $\mathcal{v} \in \mathfrak{f}^{-1}(\mathbb{H}), \mathcal{u} \in \mathfrak{f}^{-1}(\mathbb{D})$ and $\mathfrak{f}^{-1}(\mathbb{H}) \cap \mathfrak{f}^{-1}(\mathbb{D}) = \emptyset$.

Then $\mathbb{X}$ is $T_{\mathcal{P}2}$_space. ■