جامعة تكريت – كلية التربية للبنات –قسم الرياضيات

المرحلة : الرابعة

المادة: التبولوجيا العامة

عنوان المحاضرة : الترابط في الفضاء التبولوجية

مدرس المادة : أ .د. رنا بهجت ياسين

الايميل الجامعي : Zain 2016@ tu.edu.iq

In this section, study the notions of Connectedness

if $\mathbb{X}$ can be expressed as the union of two disjoint non-empty open subsets of $\mathbb{X}$, otherwise $\mathbb{X}$ is said to be connected space.

Definition (19)

Let (M,τ)be topological, for any two non-empty disjoint open subsets $H\ and\ D$ of M are said to be separated sets briefly ($Sepa$) if $H \cap cl(D) = \emptyset$ and $D \cap cl(H) = \emptyset$.

Remarks

1-Every two disjoint open (closed) subsets of any space are separated space.

2- Let $\mathbb{H}$ and $\mathbb{D}$ be separated subsets of $\mathbb{X}$, then if $\mathbb{D} \subseteq \mathbb{H}$ and $\mathcal{S} \subseteq \mathcal{K}$, then $\mathbb{D}$ and $\mathcal{S}$ are also separated

Theorem

Every two open (closed) subsets of $\mathbb{X}$ are separated, iff they are disjoint.

Proof.

Any two $\mathbb{H}, \mathbb{D}$ are separated sets, then two open sets and $\mathrm{cl}(\mathbb{H}) = \mathbb{H}$, $\mathrm{cl}\,(\mathbb{D}) = \mathbb{D}$, so that $\mathbb{H} \cap \mathrm{cl}(\mathbb{D}) = \emptyset$ and $\mathbb{D} \cap \mathrm{cl}(\mathbb{H}) = \emptyset$, then $\mathbb{H} \cap \mathbb{D} = \emptyset$.

Conversely,

If $\mathbb{H}, \mathbb{D}$ are both disjoint open, then $\mathbb{H}^c$and $\mathbb{D}^c$ are both closed, so that; $\mathbb{H} \cap \mathbb{D} = \emptyset \to \mathbb{H} \subseteq \mathbb{D}^c$ and $\mathbb{D} \subseteq \mathbb{H}^c$, also $\mathrm{cl}_{\mathcal{P}}(\mathbb{D}^c) = \mathbb{D}^c$and $\mathrm{cl}\,(\mathbb{H}^c) = \mathbb{H}^c$.

We get $\mathrm{cl}(\mathbb{D}) \subseteq \mathrm{cl}\,(\mathbb{H}^c) = \mathbb{H}^c$and $\mathrm{cl}(\mathbb{H}) \subseteq \mathrm{cl}(\mathbb{D}^c) = \mathbb{D}^c$.

Hence $\mathbb{H}$ and $\mathbb{D}$ are separated sets, so that $\mathbb{H} \cap \mathrm{cl}(\mathbb{D}) = \emptyset$, $\mathbb{D} \cap \mathrm{cl}(\mathbb{H}) = \emptyset$.■

Remark (20)

The space (M,τ) is said to be disconnected $(Dconnt)$ if and only if it is the union of two non _empty separated sets. That is there exist two non-empty disjoint separated sets D,H such that $H \cap cl(D) = \emptyset$ and $D \cap cl(H) = \emptyset$ and $H \cup D = E$. We say that E is said to be Connected if and only if it is not $Dconnt$.

Theorem (21)

The space (M,τ) is $connt.$ if and only if M cannot be expressed as the union of two disjoint non_ empty open set of (M,τ).

Proof.

Suppose that (M,τ) is $Connt$ space and $M = H \cup D$, where $H \neq \emptyset \neq D$ are closed sets, then $H = D^c and\ D = H^c$ (because $M = H \cup D$ and $H \cap D = \emptyset$). Since $H = D^c$and D is closed set and $D = H^c$and H is closed set, we get $H \in \tau$ and $H \in \tau$ Then (M,τ) is $Dconnt$ this is contradiction. Hence $M \neq H \cup D$, where H,D are closed sets, $H \cap D = \emptyset$ and $H \neq \emptyset \neq D$.

In a similar way we prove the opposite direction.■

Proposition(22)

The space $(M,\tau)$is $Connt$ if and only if any subsets of M which are both CO sets are M and $\emptyset$.

Proof. Follows from theorem (21). ■

Proposition (23)

Each two non-empty A, B subsets of (M,τ) are $Sepa$. If A is $Connt$set of (M,τ) with $A \subseteq B \subseteq cl(A)$. Then B is $Connt$.

Proof.

Claim B is not $Connt$. Then there be $Sepa$. set H and D such that $B = H \cup D$, whenever $H \cap cl(D) = cl(H \cap D) = \emptyset$ for each H and D are non_ empty. By Proposition 22,we obtain $A \subseteq H$ or $A \subseteq D$.

1. let $A \subseteq H$ then $cl(A) \subseteq cl(H)$ and $cl(H) \cap D = \emptyset$. By hypothesis, $D \subseteq B \subseteq cl(A)$and $cl(A) \cap D = D = \emptyset$, this is contradiction
2. let $A \subseteq D$ similarly we get H is empty , this is contradiction [because $H \neq \emptyset$].

Therefore B is $Connt$■

Proposition (24)

If $H\ and\ D$ are $Sepa$ sets of (M, τ) and G is $Connt$ set of (M, τ) such that $G \subseteq H \cup D$. Then $G \subseteq H$ or $G \subseteq D$.

Proof .

Assam that G is not $Connt$set and $G = (G \cap H) \cup (G \cap D)$, suppose that $(G \cap H) \cap cl(G \cap D) \subseteq H \cap cl(D) = \emptyset$ and $(G \cap H) \cap cl(G \cap D) \subseteq D \cap cl(H) = \emptyset$, then G is not $Connt$ only if $(G \cap H) = \emptyset\ and\ (G \cap D) = \emptyset$,which is a contradiction. Hence either $(G \cap H) = \emptyset\ or\ (G \cap D) = \emptyset$. This $G \subseteq H$ or $G \subseteq D$. ■

Proposition(25)

Each two non-empty A, B subsets of a (M, τ) ,if A and B are $Connt$ and $A \cap B \neq \emptyset$ in(M, τ). Then $A \cup B$ is $Connt$.

Proof.

Let $A \cup B$ be not $Connt$, then $\exists\ H, D$ are $Sepa$ in M such that

$A \cup B = H \cup D$. Then $A \subseteq H \cup D$, by Proposition (24) implies that

$A \subseteq H\ or\ A \subseteq D$ and $B \subseteq H\ or\ B \subseteq D$.

If $A \subseteq H$ and $B \subseteq H$, then $A \cup B = H$ and $D = \emptyset,$ this is contradiction, so $A \subseteq H$ and $B \subseteq D$.Similarly $A \subseteq D$ and $B \subseteq H$.

Hence $cl(A) \cap B \subseteq cl(H) \cap D = \emptyset$ and$cl(B) \cap A \subseteq cl(H) \cap D = \emptyset$

Thus $Aand\ B$ is $Sepa$ in M, this is contradiction. Hence $A \cup B$ is $Connt$. ■