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المرحلة : الرابعة

المادة: التبولوجيا العامة

عنوان المحاضرة : بعض خواص الترابط في الفضاء التبولوجية

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Proposition (26)

If E is a $Connt$ subset of $M,$ then $cl(E)$ is $Connt.$

Proof.

Let E be $Connt$ subset of a (M,τ) and suppose that $cl(E)$ is $Dconnt$,it follows that there exist A and B are non_ empty sets $\ni$ $cl(A) \cap B = \emptyset = cl(B) \cap A$.So, $cl(E) = A \cup B$,then $E \subseteq cl(E) = A \cup B$, Since E is a $Connt$ thus $E \subseteq A \cup B$ and $E \subseteq A$ or $E \subseteq B$. If, $E \subseteq A$, $cl(E) \subseteq cl(A)$ then $cl(E) \cap B \subseteq cl(A) \cap B$=$\emptyset$--(1). Since $cl(E) = A \cup B \rightarrow B \subseteq cl(E) \rightarrow cl(E) \cap B = B \rightarrow B = \emptyset$.For $cl(E) \cap B = \emptyset$ (from(1)) and H in $M \ni cl(E) = H \cup D$.Since $E = (H \cap E) \cup (D \cap E)$ and $cl(H \cap E) \subseteq cl(H)$ and $cl(D \cap E) \subseteq cl(D)$and $H \cap D = \emptyset$,then $cl(D \cap E) \cap H$=$\emptyset$, we get $cl(D \cap E) \cap (H \cap E)$= $\emptyset$.

Similarly, $cl(H \cap E) \cap (D \cap E)$= $\emptyset$. Therefore, E is $Connt$this contradiction for $B \neq \emptyset$. Similarly, $E \subseteq B \rightarrow A = \emptyset$. Hence, if E is $Connt$, then $cl(E)$ is $Connt$. ■

Proposition(27)

Let $f\colon (M,\tau) \rightarrow (N,\sigma)$ be a surjection $Cont.$ map. If $(M,\tau)$is $Connt$, then $(N,\sigma)$is $Connt$ too.

Proof.

Let (N,σ)be not $Connt$ and $M = H \cap D$,where H, D are separated non_ empty $open$ sets in (N,σ)Thus $M = f^{-1}(H) \cup f^{-1}(D)$

where $f^{-1}(H), f^{-1}(D)$ are $Sepa$ non_ empty $open$ sets in N this is contradiction, then $(N,\sigma)$is $Connt$. ■

Remarks (2.1.11)

1. $\mathbb{X}$ is connt. set, iff it is not the union of two non-empty separated sets.
2. If $\mathbb{X}$ is the union of two disjoint non_ empty $\mathcal{P}$_ open sub sets then $\mathbb{X}$ is DDconnt.
3. If E is connt. set of $\mathbb{X}$ and $\mathbb{H}$, $\mathbb{D}$ are $\text{Separ}_{\mathcal{P}}$ sets of $\mathbb{X}$ with $E \subseteq \mathbb{H} \cup \mathbb{D}$,then either $E \subseteq \mathbb{H}$ or $E \subseteq \mathbb{D}$.
4. If E subset of $\mathbb{X}$ is a connt. . Then $\text{cl}_{\mathcal{P}}(E)$ is connt.

We know that if $\mathbb{H}$ and $\mathbb{D}$ are connt. sets, then $\mathbb{H} \cup \mathbb{D}$ is Dconnt. set, but by adding some condition we can prove that connt. sets by the following theorem.

Theorem (2.1.12)

If $\mathbb{H}$ and $\mathbb{D}$ is connt. sets, such that $\mathbb{H} \cap \mathbb{D} \neq \emptyset$. Then $\mathbb{H} \cup \mathbb{D}$

is connt. set.

Proof.

Suppose that $\mathbb{H}, \mathbb{D} \subseteq \mathbb{X} \ni \mathbb{H} \cap \mathbb{D} \neq \emptyset$ and $\mathbb{H}, \mathbb{D}$ are connt. and

$\mathbb{H} \cup \mathbb{D}$ is connt . If $\mathbb{X}, \mathbb{Y}$ are two disjoint non_ empty open

sets and $\mathbb{X}, \mathbb{Y} \in \tau$, then $\mathbb{H} \cup \mathbb{D} = \mathbb{X} \cup \mathbb{Y}$.

So, $\mathbb{H} \subseteq \mathbb{H} \cup \mathbb{D} \rightarrow \mathbb{H} \subseteq \mathbb{X} \cap \mathbb{Y} \rightarrow \mathbb{H} \subseteq \mathbb{X}$ or $\mathbb{H} \subseteq \mathbb{Y}$ (because $\mathbb{H}$ isconnt.). Also

$\mathbb{D} \subseteq \mathbb{H} \cup \mathbb{D} \rightarrow \mathbb{D} \subseteq \mathbb{X} \cap \mathbb{Y} \rightarrow \mathbb{D} \subseteq \mathbb{X}$ or $\mathbb{D} \subseteq \mathbb{Y}$(because $\mathbb{D}$ is connt.).

Now, either $\mathbb{H} \subseteq \mathbb{X} \wedge \mathbb{D} \subseteq \mathbb{X} \rightarrow \mathbb{H} \cup \mathbb{D} \subseteq \mathbb{X} \rightarrow \mathbb{Y} = \emptyset$,this is contradiction.

or $\mathbb{H} \subseteq \mathbb{Y} \wedge \mathbb{D} \subseteq \mathbb{Y} \rightarrow \mathbb{H} \cup \mathbb{D} \subseteq \mathbb{Y} \rightarrow \mathbb{X} = \emptyset$, this is contradiction.

or $\mathbb{H} \subseteq \mathbb{Y} \wedge \mathbb{D} \subseteq \mathbb{X} \rightarrow \mathbb{H} \cap \mathbb{D} \subseteq \mathbb{X} \cap \mathbb{Y} = \emptyset \rightarrow \mathbb{X} \cap \mathbb{Y} = \emptyset$,

this is contradiction.

or $\mathbb{H} \subseteq \mathbb{X} \wedge \mathbb{D} \subseteq \mathbb{Y} \rightarrow \mathbb{H} \cap \mathbb{D} \subseteq \mathbb{X} \cap \mathbb{Y} = \emptyset \rightarrow \mathbb{X} \cap \mathbb{Y} = \emptyset$,

this is contradiction. Hence $\mathbb{H} \cup \mathbb{D}$ is connt . ∎

And by generalizing the above theorem to any family of connt . sets, we obtain the following theorem.

Proportion

The union of any family of $\mathrm{Connt}_{\mathcal{P}}$ sets have non_ empty intersection connt. set.

Proof.

Let $\{\mathcal{M}_{\mathrm{i}} : \mathrm{i} \in \mathbb{N}\}$ be non-empty of connt. subset of $\mathbb{X}$ and $\bigcup_{\mathrm{i}\in\Lambda} \mathcal{M}_{\mathrm{i}}$ is Dconnt., then $\bigcup_{\mathrm{i}\in\Lambda} \mathcal{M} = \mathbb{H} \cup \mathbb{D}$, where $\mathbb{H}$ and $\mathbb{D}$ are separated sets in $\mathbb{X}$. Since $\bigcap_{\mathrm{i}\in\Lambda} \mathcal{M}_{\mathrm{i}} \neq \emptyset$, then $\mathrm{x} \in \bigcap_{\mathrm{i}\in\Lambda} \mathcal{M}_{\mathrm{i}}$.

Since $\mathrm{x} \in \bigcup_{\mathrm{i}\in\Lambda} \mathcal{M}_{\mathrm{i}}$ either $\mathrm{x} \in \mathbb{H}$ or $\mathrm{x} \in \mathbb{D}$, if $\mathrm{x} \in \mathbb{H} \wedge \mathrm{x} \in \mathcal{M}_{\mathrm{i}}, \forall \mathrm{i} \in \mathbb{N}$. By (Remarks $\mathcal{M}_{\mathrm{i}} \subseteq \mathbb{H}$ or $\mathcal{M}_{\mathrm{i}} \subseteq \mathbb{D}$. Since $\mathbb{H} \cap \mathbb{D} = \emptyset$.

Then $\bigcup_{\mathrm{i}\in\Lambda} \mathcal{M}_{\mathrm{i}} \subseteq \mathcal{H}$(because $\mathcal{M}_{\mathrm{i}} \subseteq \mathbb{H}$ for all $\mathrm{i} \in \mathbb{Z}$), that leads to $\mathbb{D}$ is empty , this is a contradiction.

By similar discussion $\mathbb{H}$ is also empty and this is a contradiction.

Then $\bigcup_{\mathrm{i}\in\Lambda} \mathcal{M}_{\mathrm{i},}$ is connt . set. ■