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In this section, we review the definition of open and closed functions and propositions about this subject.

Definitions (9)

The function $f:(M,\tau)\to(N,\sigma)$ is called :

1. open function (OM) if, $f(A)\in O(\breve{M},\breve{X})$, for each $A\in(M,\tau)$.
2. closed function (CM) if, $f(B)\in C(\breve{M},\breve{X})$, for each B is closed set.

Proposition (10)

A surjective map $f:(M,\tau)\to(N,\sigma)$ is OM if and only if f is CM.

Proof.

Let f be CM map and $A\subseteq M, A\in(M,\tau)$thus A^c is a closed set. Since f is a CM, then $f(A^c)$ is a closed set in $\left(\breve{M},I\left(\breve{X}\right)\right)$.Therefore $f(A^c)^c$ is a open set, so $f(A^c)^c=f(A)$ (because f be surjective). Hence $f(A)$ is a open set in (M,τ)

So f is OM. In similar way we can prove that (2) ■

Proposition (11)

The function $f:(M,\tau)\to(N,\sigma)$ is a OM if and only if $f\left(int(A)\right)\subseteq int(f(A))$ for each $A\subseteq M$.

Proof.

Let f be $open$ function then for any $H\in I\to f(H)\in(M,\tau)$ and $A\subseteq M$, since $int\ (A)\subseteq A\to f(int\ A)\subseteq f(A)\ \to$
$int\ (f(int\ A))\subseteq int\ (f(A))$, then $int\ A\in I\ \to f(int\ A)\in\ I(\breve{X})$.
For f is $open$ and $int\ A$ is open set, hence

$f(int\ A) = int\ \big(f(int\ A)\big)$[because $B^{\circ} = B \to \forall B\ is\ open$]

Then $f(int\ A) \subseteq int\ (f(A))$.

Conversely,

For each $A \subseteq M$ which that $f(int\ A) \subseteq int\ (f(A))$, if $H \in I$ be arbitrary so that $int\ (H) = H \to f(int\ H) = f(H)$ But $f(int\ H) \subseteq int\ f(H)$,combining these two results, $f(H) \subseteq int\ f(H)$ also $int\ f(H) \subseteq f(H)$ then $int\ f(H)$ $=f(H)$, thus $H \in I \to f(H) \in (M,\tau)$ Then f is OM. ■

<u>Proposition(12)</u>

Let $f:(M,\tau) \to (N,\sigma)$ be a CM function if and only if $cl\ (f(A)) \subseteq f(cl\ (A))$, for each $A \subseteq M$.

Proof.

Suppose that f a CM, then $f\left(cl_{N_P}(A)\right)$ is a closed set in (M,τ) for each $A \subseteq M$. Since $cl\ (A)$ is a closed set in (M,τ) so $A \subseteq cl\ (A) \to f(A) \subseteq f(cl\ (A))$.Thus $f(cl(A))$ is a $closed$ set containing $f(A)$, we get $cl(f(A)) \subseteq f(cl(A))$.

Conversely,

Let $cl(f(A)) \subseteq f(cl(A))$, for each $A \subseteq M$ and f be a closed set in (M,τ) ,then $cl\ (A) = A$ and $cl\ f(A) = f(A)$, thus $f(A)$ is a closed set in (M,τ) Hence f is a CM. ■

<u>Propositions (13)</u>

Let $f:(M,\tau) \to (N,\sigma)$be a bijective function, then:

1. f is a OM if and only if f^{-1} is a Con.
2. f is a CM if and only if f^{-1} is a Con.

Proof.

Let f be a OM and $A \subseteq M$, then $f(A)$ is open set in (M,τ), $(f^{-1})^{-1}(A) = f(A) is \ open$ set in (M,τ) . Then f^{-1} is a Con.

Conversely,

For each $A \subseteq M$,since f is bijective then $(f^{-1})^{-1}(A)= f(A)$. Let A is open set in $\big(M, I(X)\big)$, we get $(f^{-1})^{-1}(A)$ is an open set in (M,τ) (because f^{-1} is a Con) and since f be bijective, then $f(A)$is a open set in (M,τ) Hence f is a OM. In similar way we can prove that (2). ■

Remark

Let (M,τ) is a T_2_space, If $(N,\sigma)$is a compact subspace of (M,τ) then $(N,\sigma)$is closed.

Proposition

If $\mathbb{A}$ is compact subset of a T_2 −space (M,τ) , then $\mathbb{A}$ is closed set

Proof.

Suppose that $\mathbb{A}^c$is closed set and $u \in \mathbb{A}^c$, since $\mathbb{A}$ is a compact subset of a T_2_space, if $u \notin \mathbb{A}$, then $\exists\ G \in (M,\tau)$ such that $u \in G \subseteq \mathbb{A}^c$.

Therefore $\mathbb{A}^c = \cup \{G: u \in \mathbb{A}^c\}$,thus $\mathbb{A}^c$is closed set , as it is the union of closed sets . Then $\mathbb{A}$ is closed set ■

Proposition

The image of a $\mathcal{N}_{\mathcal{P}}$_compact space under a cont. map is compact.

Proof.

If $((M,\tau)$ is compact to any $\mathcal{N}\_$ topological space (N,σ)Let $f:(M,\tau)\to(N,\sigma)$ be cont. map and $\{\mathrm{G_i}: \mathrm{i} \in \wedge\}$ be an open set cover of (N,σ)

Then $\{\mathfrak{f}^{-1}(\mathrm{G_i}): \mathrm{i} \in \wedge\}$ is open set cover of (M,τ) and has a finite sub cover $\{\mathfrak{f}^{-1}(\mathrm{G_i}): \mathrm{i} = 1,2,3,\ldots,\mathrm{n}\}$[because $\mathfrak{f}$ is cont. and $\mathcal{M}$ is compact].

Whenever $\mathcal{M} = \{\cup\, \mathfrak{f}^{-1}(\mathrm{G_i}): \mathrm{i} \in \wedge\} \to \{\cup\, (\mathrm{G_i}): \mathrm{i} \in \wedge\} = \mathfrak{f}\,(\mathcal{M}) = \breve{\mathcal{M}}$.

Thus $\{\mathrm{G_1}, \mathrm{G_2}, \mathrm{G_3}, \ldots \mathrm{G_n}\}$ is finite sub cover of $\{\mathrm{G_i}: \mathrm{i} \in \wedge\}$ for (N,σ)Hence $(N,\sigma)$is compact